DOCUMENT RESUME

ED 160 453 SE 025 102

AUTHOR	Allen, Frank B.; And Others
TITLE	Geometry with Coordinates, Student's Text, Part II, Unit 48. Revised Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	62
NOTE	514p.; For related documents, see SE 025 101-104
EDRS PRICE	MF-$1.00 HC-$27.45 Plus Postage.
DESCRIPTORS	*Analytic Geometry; Curriculum; *Geometry; *Instructional Materials; Mathematics Education; Secondary Education; *Secondary School Mathematics; *Textbooks
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This is part two of a two-part SMSG geometry text for high school students. One of the goals of the text is the development of analytic geometry hand-in-hand with synthetic geometry. The authors emphasize that both are deductive systems and that it is useful to have more than one mode of attack in solving problems. The text begins the development of geometry synthetically and teaches the method of synthetic proof, then leads quickly to the use of coordinate systems in the remainder of the work. Chapter topics include: coordinates in a plane; perpendicularity, parallelism, and coordinates in space; directed segments and vectors; polygons and polyhedrons; and circles and spheres. (MN)

Reproductions supplied by EDRS are the best that can be made from the original document.

STUDENT'S TEXT UNIT NO. 48

ED160453

U S DEPARTMENT OF HEALTH.
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY

SMSG

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC) AND USERS OF THE ERIC SYSTEM."

GEOMETRY WITH COORDINATES

PART II

SE 025 102

SCHOOL MATHEMATICS STUDY GROUP

YALE UNIVERSITY PRESS

2

Geometry with Coordinates

Student's Text, Part II

REVISED EDITION

Prepared under the supervision of a
Panel on Sample Textbooks
of the School Mathematics Study Group:

Frank B. Allen	Lyons Township High School
Edwin C. Douglas	Taft School
Donald E. Richmond	Williams College
Charles E. Rickart	Yale University
Robert A. Rosenbaum	Wesleyan University
Henry Swain	New Trier Township High Scho
Robert J. Walker	Cornell University

New Haven and London, Yale University Press

Copyright © 1962 by The Board of Trustees
of the Leland Stanford Junior University.
Printed in the United States of America.

All rights reserved. This book may not
be reproduced in whole or in part, in
any form, without written permission from
the publishers.

Financial support for the School Mathematics
Study Group has been provided by the
National Science Foundation.

CONTENTS

Chapter

8. COORDINATES IN A PLANE 505
- 8-1. Introduction 505
- 8-2. A Coordinate System in a Plane 509
- 8-3. Basic Theorems 519
- 8-4. The Set-Builder Notation 530
- 8-5. Composite Conditions 533
- 8-6. Equations and Inequalities 538
- 8-7. Finding the Coordinates of the Points of a Line 543
- 8-8. Slope 552
- 8-9. Other Equations for Lines 567
- 8-10. Perpendicular Lines 572
- 8-11. Parallelograms 577
- 8-12. Using Coordinates in Proofs 580
- 8-13. Parallelograms Continued 585
- 8-14. Trapezoids 593
- 8-15. Concurrent Lines 595
- 8-16. Summary 602
- Review Problems 603

9. PERPENDICULARITY, PARALLELISM, AND COORDINATES IN SPACE 607
- 9-1. Introduction 607
- 9-2. Perpendicularity Relations 609
- 9-3. Parallel Relations 617
- 9-4. Relations Involving Perpendicularity and Parallelism 623
- 9-5. Dihedral Angles 633
- 9-6. Coordinate Systems in Space 641
- 9-7. The Distance Formula in Space 651
- 9-8. Parametric Equations of a Line in Space . . 658
- 9-9. Equation of a Plane 662
- 9-10. Summary 670
- Vocabulary 672
- Review Problems 673

Review Problems, Chapters 6 to 9 675

Chapter

10. DIRECTED SEGMENTS AND VECTORS 683
10-1. Introduction 683
10-2. Directed Segments 685
10-3. Vectors 703
10-4. The Two Fundamental Theorems 710
10-5. Geometrical Application of Vectors 714
10-6. The Scalar Product of Two Vectors 717
10-7. Summary. 722

11. POLYGONS AND POLYHEDRONS 727
11-1. Introduction 727
11-2. Polygonal-Regions 728
11-3. Sum of the Measures of the Angles of a Convex Polygon 735
11-4. Area 743
11-5. Areas of Triangles and Quadrilaterals . . . 753
11-6. Area Relations 765
11-7. Relations in Similar Polygons 773
11-8. Regular Polygons 779
11-9. Polyhedrons 782
11-10. Polyhedral Angles 788
11-11. Prisms 796
11-12. Pyramids 803
11-13. Summary 809
Review Problems 811

12. CIRCLES AND SPHERES 819
12-1. Basic Definitions 819
12-2. Tangent Lines 829
12-3. Tangent Planes 841
12-4. Arcs of Circles 847
12-5. Lengths of Tangent and Secant Segments . . . 869
12-6. The Circumference of a Circle; the Number π 887
12-7. Area of a Circle 893
12-8. Lengths of Arcs. Areas of Sectors 899
12-9. Inscribed and Circumscribed Circles 905
12-10. Summary 910

Review Problems 910

Review Problems, Chapters 10 to 12 913

Appendix V. How to Draw Pictures of Space Figures 921

Appendix VII. Surface Area and Volume. 929

Appendix VIII. How Eratosthenes Measured the Earth 941

Appendix IX. Rigid Motion 943

Appendix X. Trigonometry 963

Appendix XI. Vectors in Space 973

Appendix XII. Applications of Geometric Theory to the Use of Straightedge and Compasses in Drawing Pictures of Plane Figures 975

The Meaning and Use of Symbols 983

The Greek Alphabet. 987

List of Chapters and Postulates 989

List of Theorems and Corollaries 991

Index

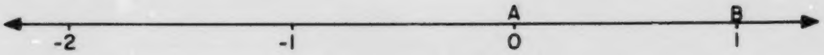
List of Writers

Chapter 8

COORDINATES IN A PLANE

8-1. Introduction.

In connection with our discussion of distance we introduced the idea of a coordinate system on a line. A coordinate system on a line is determined by any pair of points on it; with one point of this pair designated as the origin and the other designated as the unit-point. A coordinate system on a line is a one-to-one correspondence between the set of all real numbers and the set of all points in the line, such that the coordinates, i.e., the numbers associated with the points, can be used to determine distances between points.



Problem Set 8-1

1. On line $\overleftrightarrow{AB}$ assume a coordinate system which assigns the coordinate 0 to A and 1 to B . P is a point on $\overleftrightarrow{AB}$ with coordinate x . For each listed condition plot the set of all points P determined by that condition.

   (a) $x = 5$.
   (b) $x = -3$.
   (c) $x = 3AB$.
   (d) $x = 4AB$.
   (e) $x = \frac{1}{2} AB$.
   (f) $x = t \cdot AB$ and t is an element of $\{1, 4, 0, \frac{1}{2}\}$.
   (g) $x = k \cdot AB$; $k \leq 1$.
   (h) $x = k \cdot AB$; $k \geq 0$.
   (i) $x = k \cdot AB$; $0 \leq k < 1$.
   (j) $x = k \cdot AB$; $k > 0$.

2. A coordinate system on $\overleftrightarrow{AB}$ assigns coordinates 0, 1, x to points A, B, P respectively. Plot the set of all P such that x satisfies the given conditions.

   (a) $x > 0$ and x is an integer. Describe the set.
   (b) $x < 0$ and x is real.
   (c) $-2 < x < 5$; x is an integer. How many points belong to this set?
   (d) $-3 < x \leq 1$; x is real. How many points belong to this set?
   (e) $-5 \leq x \leq -1$; x is real.
   (f) Using mathematical terms describe the point sets in (b), (d), (e).

3. If A, B, C, P are on ray $\overrightarrow{AP}$ and have respective coordinates 0, 1, 3, x, what is the value or values of x determined by each of the following conditions?

   (a) $AP = 2AC$.
   (b) $AP = 5AC$.
   (c) $AP = k \cdot AC$.
   (d) $BP = 3BC$.
   (e) $BP = k \cdot BC$.
   (f) $BP = 2AC$.

4. Suppose a coordinate system on a line m is given. If P and Q are points in m with coordinates p and q respectively, find the distance from P to Q , if:

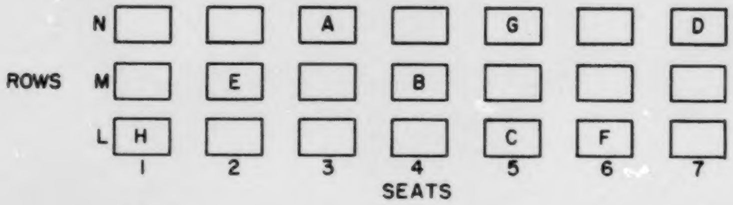
   (a) $p = 5$, $q = 8$.
   (b) $p = -7$, $q = -8$.
   (c) $p = 3$, $q = -5$.
   (d) $p = -9$, $q = 4$.
   (e) $p = r - 3$, $q = r + 3$.
   (f) $p = r + 9$, $q = r + 12$.
   (g) $p = a$, $q = -a$, $a > 0$.
   *(h) $p = a$, $q = b$.

5. Suppose a coordinate system is established on line m , and P and Q are points on m with coordinates p and q respectively. If P, T_1, M, T_2, Q are collinear in that order and represent the midpoint and trisection points of $\overline{PQ}$, find the coordinates of M, T_1, T_2 in the following. Record your results for each problem on a separate number line. (Refer to Theorem 3-6.)

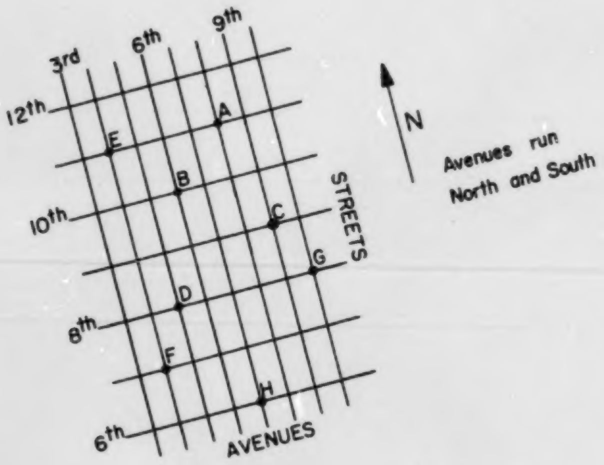
   (a) $p = 3$, $q = 12$.
   (b) $p = -10$, $q = -1$.
   (c) $p = -2$, $q = 13$.
   (d) $p = a$, $q = b$, $b > a$.
   (e) $p = r + a$, $q = r - a$, $a < 0$
   (f) $p = (r + b) - 2$.
   $q = (r + b) + 4$.

6. In each of the following problems indicate the location of the objects lettered from A through H by using either a pair or a triple of symbols.

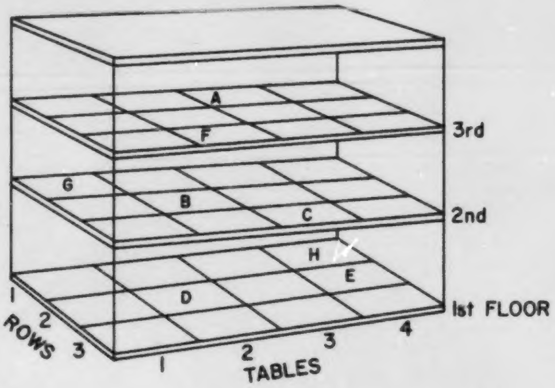
(a) Seats in an auditorium.


ROWS
N
M
L
A
G
D
E
B
H
C
F
1
2
3
4
5
6
7
SEATS

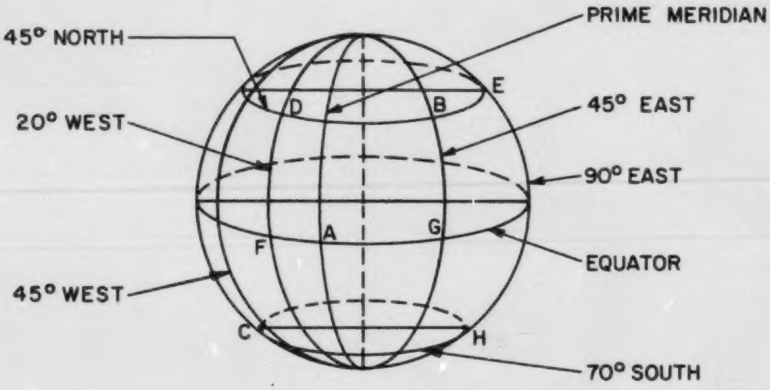
(b) Houses at the intersection of streets and avenues.


3rd
6th
9th
12th
10th
8th
6th
A
B
C
D
E
F
G
H
STREETS
AVENUES
N
Avenues run North and South

(c) Tables in rows in the floors in a store.


A
F
3rd
G
B
C
2nd
H
E
D
Ist FLOOR
1
2
3
ROWS
1
2
3
4
TABLES

(d) Points on the surface of the earth.


PRIME MERIDIAN
45° NORTH
E
D
B
20° WEST
45° EAST
90° EAST
A
G
F
EQUATOR
45° WEST
C
H
70° SOUTH

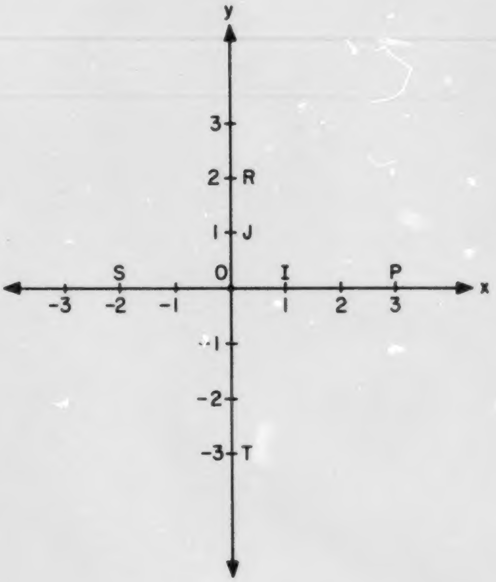
(e) Using the data of Part (d), indicate the position of airplanes which are above each of the listed points. Assume the one above A has an elevation of 5000 ft., and that the elevation of each one from A to H is 200 ft. more than the preceding one.

8-2. A Coordinate System in a Plane.

Suppose that a plane is given and, until further notice, that all points and lines under consideration lie in this plane. Suppose further that a unit-pair of points $\{A, A'\}$, as discussed in Chapter 3, is given. All distances are to be considered as measures of distances relative to this unit-pair.

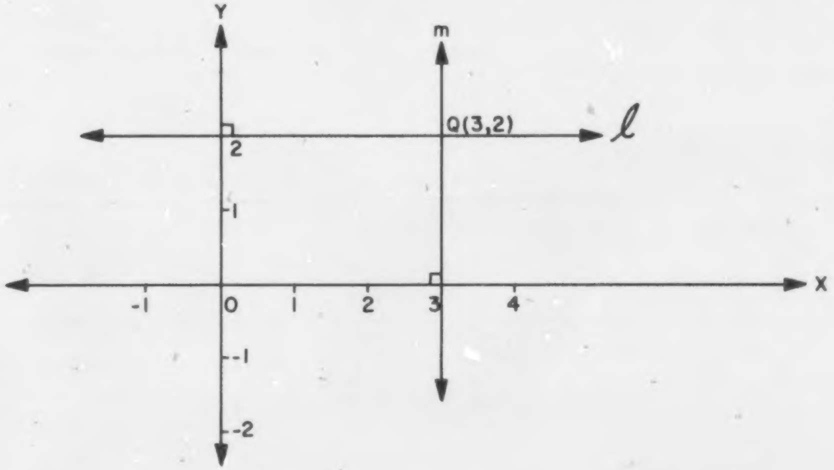
Let $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$ be any two perpendicular lines with O their point of intersection. Let I and J be points in $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$, respectively, such that $OI = 1 = OJ$. There is a coordinate system on $\overleftrightarrow{OX}$ with the point O as origin and the point I as unit point. We call this the x-coordinate system and the coordinate in this system of a point of $\overleftrightarrow{OX}$ its x-coordinate. Similarly O and J are the origin and unit point of a coordinate system on $\overleftrightarrow{OY}$. We call this system the y-coordinate system and the coordinate in this system of a point on $\overleftrightarrow{OY}$ its y-coordinate.

Thus in the diagram I and J are the unit points of their respective coordinate systems. Point O is the origin of both the x- and y-coordinate systems. The coordinate of point P is 3 with respect to the x-coordinate system and the coordinate of R with respect to the y-coordinate system is 2 . Name the coordinates of points S and T . Is it necessary to specify the coordinate system in each case? Why?

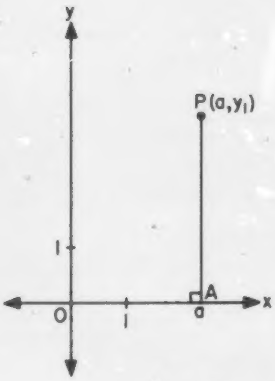


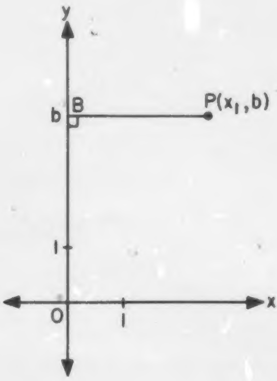
The line $\overleftrightarrow{OX}$ is called the *x-axis* and $\overleftrightarrow{OY}$ is called the *y-axis*; their point of intersection, O, is called the *origin*; and the plane determined by them is called the *xy-plane*. Because of the way these axes usually are shown in pictures on a chalkboard, it is customary to call lines parallel to the x-axis *horizontal lines*, and lines parallel to the y-axis *vertical lines*. It is customary to think of I as lying to the right of the origin and of J as lying above the origin. This means, then, that the points on the x-axis with positive coordinates lie to the right of the origin, while the points on the x-axis with negative coordinates lie to the left of the origin. Where do the points on the y-axis with positive coordinates lie? Where do the points on the y-axis with negative coordinates lie?

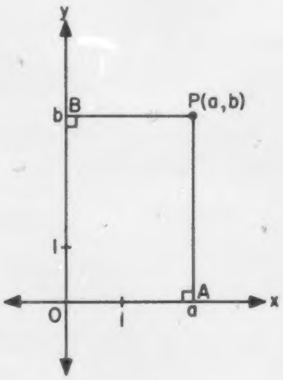
We are now ready to define a coordinate system in the xy-plane which is determined by the x- and the y-coordinate systems. We consider a particular point first. Suppose that Q is a point, that the vertical line through Q cuts the x-axis in the point whose x-coordinate is 3, and the horizontal line through Q cuts the y-axis in the point whose y-coordinate is 2. We say in this case that the x-coordinate of Q is 3, that the y-coordinate of Q is 2, and we call the ordered pair of numbers $(3,2)$ the coordinates of Q.



We are now ready for the general case. Let P be any point in the xy-plane. From our previous work we know that there is exactly one line through P perpendicular to the x-axis and exactly one line through P perpendicular to the y-axis. Why? The point P has an x-coordinate and a y-coordinate which we now define. The x-coordinate of P is the *x-coordinate* of the projection of P into the x-axis and the *y-coordinate* of P is the y-coordinate of the projection of P into the y-axis. We sometimes call the x-coordinate of P and the y-coordinate of P *the coordinates of P*. The coordinates of the y-coordinate of P are considered an ordered pair of real numbers in which the x-coordinate is the first number of the pair and the y-coordinate is the second. If the x-coordinate of P is a and the y-coordinate of P is b, the coordinates of P are written as (a,b). Note that the numbers in an ordered pair need not be distinct. Thus (5,5) is an ordered pair of real numbers. Of course (8,3) and (3,8) are different ordered pairs. In fact, (a,b) = (c,d) if and only if a = c and b = d.





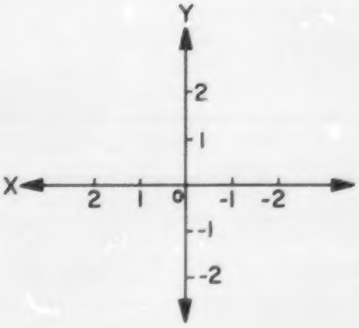


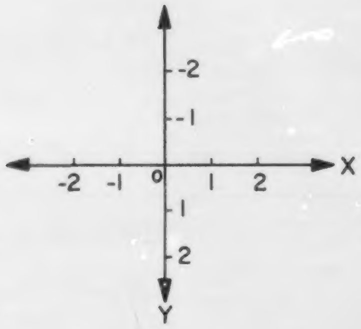
In the diagrams A is the projection of P into the x-axis and B is the projection of P into the y-axis. Thus the x-coordinate of P is a and the y-coordinate of P is b. We call the ordered number pair (a,b) the xy-coordinates of P.

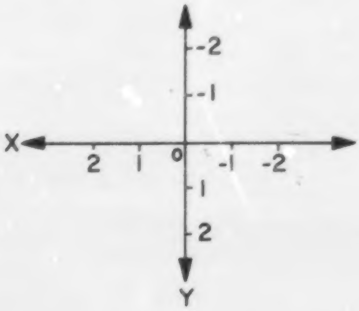
Since the projection of a point into a line is unique, it follows that there is exactly one ordered pair of real numbers assigned to each point as its coordinates. Conversely, if (a,b) is any ordered pair of real numbers there is exactly one point P in the xy-plane which has (a,b) assigned to it as its coordinates. Indeed, there is a unique vertical line through the point on the x-axis with x-coordinate a , and a unique horizontal line through the point on the y-axis with y-coordinate b . And P is the unique point in which this vertical line and this horizontal line intersect. Therefore there is a one-to-one correspondence between the set of all points in the xy-plane and the set of all ordered pairs of real numbers.

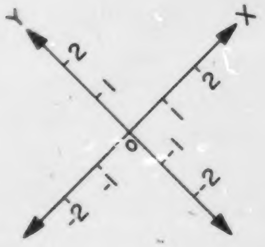
Corresponding to any three points O, I, and J , such that $\overleftrightarrow{OI} \perp \overleftrightarrow{OJ}$ and $OI = 1 = OJ$ there is a coordinate system in the xy-plane. This coordinate system is the one-to-one correspondence which we described above. Although there are many xy-coordinate systems in a plane, we usually think of only one of them in a given problem or theorem. Once a coordinate system has been set up we may use ordered pairs of real numbers as names for points. The coordinate pair of a point is a good name for a point in view of the one-to-one correspondence described above. Thus we may say that the point Q has coordinates (-2,4) , or that Q = (-2,4) . Sometimes we simply write Q(-2,4) . Occasionally we use the symbol x_A to denote the x-coordinate of the point A and the symbol y_A to denote the y-coordinate of the point A . Thus (x_A,y_A) is another name for the point A .

We have used "above," "below," "right," "left," to describe the position of a point. These words were introduced for convenience and we can get along without them if we are challenged to do so. Furthermore, there are situations (not in this book, however) in which it is convenient to take the positive part of the x-axis as extending to the left, or the positive part of the y-axis as extending downward, or some other variation.

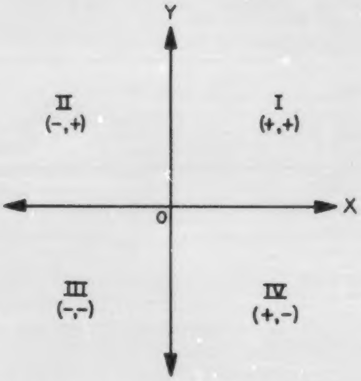






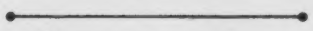


In describing the location of a point in the xy-plane it is sometimes convenient to specify the portion of the plane in which it lies. The lines $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$ form four angles. Every point in the plane lies in $\overleftrightarrow{OX}$ or in $\overleftrightarrow{OY}$ or in the interior of one of the four angles whose sides lie on $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$. The interiors of these angles are called quadrants. The first quadrant is the set of all points whose x- and y-coordinates are both positive. The second quadrant is the set of all points whose x-coordinate is negative and whose y-coordinate is positive. The third quadrant is the set of all points whose x-coordinate and y-coordinate are both negative. The fourth quadrant is the set of all points whose x-coordinate is positive and whose y-coordinate is negative. We denote these quadrants as I, II, III, IV.

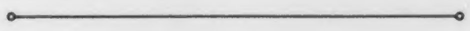


Suppose we wish to describe the location of the point $P = (5, -3)$ without using the words "right," "left," "above," "below." We might say that P is in the fourth quadrant, that it is in a vertical line which cuts the x-axis in a point 5 units from the origin, and that it is in a horizontal line which cuts the y-axis in a point which is 3 units from the origin.

In the following problems we use the words "plot" and "graph." To plot a point means to draw a picture of the axes and to mark a dot in the proper place as a picture of the point. A name for the point is frequently written beside its picture. We use the word graph to mean a set of points. To draw (or plot) a graph is to draw a picture which shows the axes and the set of points. If there are infinitely many points in a set, its graph is sometimes drawn by drawing line segments, or by shading the appropriate region. The picture of a graph always shows the axes, but they are not a part of the graph unless it is so stated. The label X is placed along the positive part of the x-axis; the label Y is placed along the positive part of the y-axis. It is usually desirable to label at least one point on $\overleftrightarrow{OX}$ other than the origin with its x-coordinate, and at least one point on $\overleftrightarrow{OY}$ other than the origin with its y-coordinate. If we wish to represent a line segment including its endpoints, we sometimes emphasize the endpoints as in the following picture.

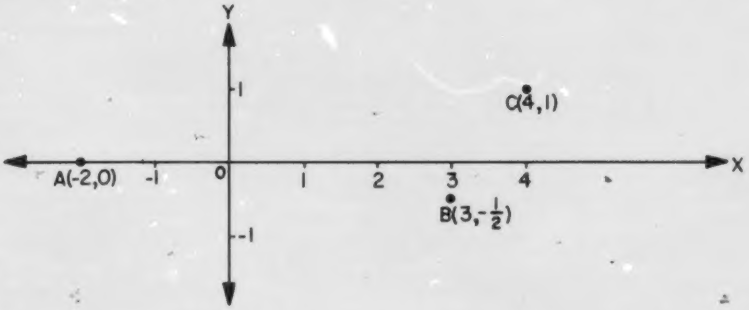


If we wish to represent a set which consists of all points of a line segment except its endpoints, we may de-emphasize the endpoints as in the following picture.

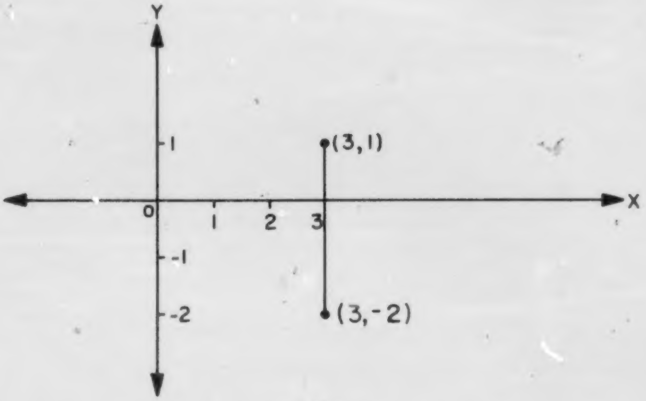


If the axes, or a portion of them, are a part of the graph, we may indicate this by making "heavier lines."

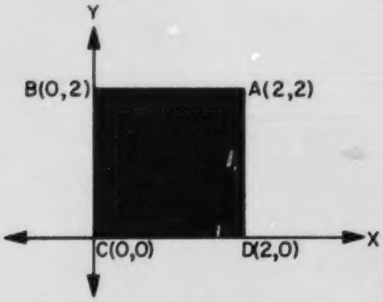
Example 1. Plot the points $A(-2,0)$, $B(3,-\frac{1}{2})$, $C(4,1)$.

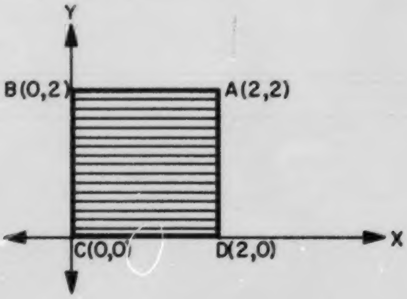


Example 2. Draw the graph of the line segment with endpoints $(3,-2)$ and $(3,1)$.

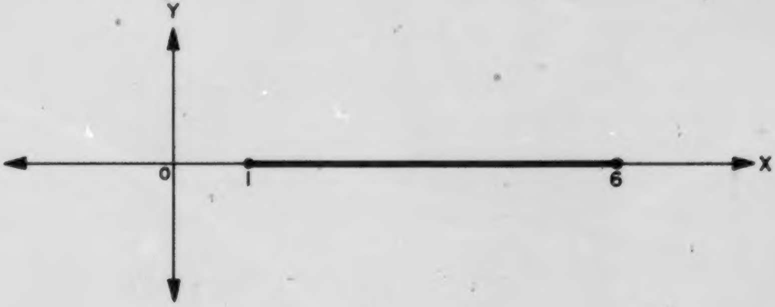


Example 3. If $A = (2,2)$, $B = (0,2)$, $C = (0,0)$, $D = (2,0)$, draw a graph of the set of all points which belong to the polygon ABCD or its interior.





Example 4. Draw a graph of the line segment whose endpoints are (1,0) and (6,0) .



Problem Set 8-2

1. Plot the points (1,0) , (3,1) , (-3,2) , (-5,4) , (4,-3.5) , (0,-2) .
2. Given $P = (5,6)$.
   - (a) If Q is the point in which the vertical line through P intersects the x-axis, what are the coordinates of Q ?
   - (b) If R is the point in which the horizontal line through P intersects the y-axis, what are the coordinates of R ?

3. Without plotting, name the quadrant in which each of the following points lies: $(3,7)$, $(-.2,-3)$, $(-6,4.3)$, $(\pi,-1)$, $(-\sqrt{2},\sqrt{3})$.

4. Explain what it means to say that our coordinate system in a plane has established a one-to-one correspondence between the set of ordered pairs of real numbers and the set of points in a plane.

5. If points P, Q, R have coordinates $(3,2)$, $(3,8)$, $(3,5)$, respectively, they are collinear in what order?

6. Describe the set of all points in a plane for which the x-coordinate is 3 ; for which the y-coordinate is -5 . Describe the intersection of these two sets.

7. Plot the set of all points (x,y) for which x and y are both integers and x and y satisfy the following conditions:
   (a) $x = 2$, $-1 \leq y \leq 5$. How many points belong to this set?
   (b) $y = -3$, $2 < x < 6$. How many points belong to this set?
   (c) $-4 < x \leq -1$, $-2 \leq y < 3$. This set contains how many points?
   (d) $0 < x \leq 2$, $-4 \leq y < 0$. This set contains how many points?

8. Plot the set of all points (x,y) in a plane satisfying the following conditions. Describe each set using mathematical terms.
   (a) $1 \leq x \leq 5$, $y = 2$.
   (b) $x \geq 1$, $y = 2$.
   (c) $x = 1$, $y \geq 2$.
   (d) $x \geq 1$.
   (e) $x = 4$, $2 < y < 5$.
   (f) $x = -3$, $y < 2$.
   (g) $x \leq 3$, $y < -1$.
   (h) $y < 0$.

*9. (a) If $A = (3,0)$ and $B = (7,0)$, what is the length of segment $\overline{AB}$? Justify your answer.
   (b) If $C = (3,4)$ and $D = (7,4)$, what is the distance CD ? Justify your answer.

10. Without plotting, arrange the following points in order from lowest to highest. Ignore the variation in their distance from the y-axis.

$$(8,6)\ ,\ (2,-3)\ ,\ (-1,-1)\ ,\ (3,0)\ ,\ (-5,4)\ ,\ (0,1)$$

11. Without plotting, arrange the following points in order from left to right. (Ignore the variation in their distance from the x-axis.)

$$(2,0)\ ,\ (-3,4)\ ,\ (0,8)\ ,\ (4,-3)\ ,\ (-\pi,6)\ ,\ (\pi,-2)$$

*12. What is the length of the segment $\overline{AB}$, given the coordinates of its endpoints as follows:

(a) $(3,8)$, $(3,-5)$.
(b) $(7,12)$, $(-6,12)$.
(c) $(4,-6)$, $(4,-10)$.
(d) $(a,$ [illegible] , (b,r) .
(e) (m,t) , $(m,5)$.

13. Describe the set of all point with coordinates (x,y) which satisfy the conditions in each of the following:

(a) $x > 0$, $y < 0$.
(b) x is positive, y is non-negative.
(c) x and y are both negative.
(d) $x > -2$,
y is any real number.
(e) x is any integer and y is any integer.
(f) x is any real number and y is any real number.

*14. Plot the points listed and in each part give the coordinates of the midpoint of the segment determined by these points.

(a) $(0,0)$, $(0,8)$.
(b) $(3,7)$, $(3,11)$.
(c) $(-5,-3)$, $(-10,-3)$.
(d) $(10,4)$, $(-10,4)$.

*15. Give the coordinates of the midpoints of the segments whose endpoints are:

(a) $(-4,a)$, $(10,a)$.
(b) $(1,a)$, $(1,b)$.
(c) $(2a,c)$, (a,c) .
(d) (x_1,y_1) , (x_2,y_1) .

16. Describe the position of the point $(-7,-8)$ without using the words "right," "left," "above," or "below."

8-3. Basic Theorems.

Now that a coordinate system for a plane has been defined, we may extend our ideas about distance to include distance between points in a plane. In doing this, it is convenient to introduce the concept of absolute value of a number.

Suppose A and B are distinct points on line ℓ and their coordinates are a and b respectively. To find AB, the distance between A and B, we take $a - b$ or $b - a$, whichever is positive. If we do not know whether $a > b$ or $b > a$, we cannot tell which is positive. Therefore to indicate that we wish to choose the one which is positive, we use the symbol $|a - b|$, read "absolute value of $a - b$." Of course we want $|a - b| = |b - a|$ and for this reason we make the following definition.

DEFINITION. If $a \geq b$ then $|a - b| = a - b$ and $|b - a| = a - b$.

Example 1. If $a = 7$ and $b = 5$, $|a - b| = 2$.
If $a = 5$ and $b = 7$, $|a - b| = 2$.

Example 2. Show that if $x > 0$ then $|-x| = x$.

Solution: $|-x| = |-x - 0|$. Since $0 > -x$,
$|-x - 0| = 0 - (-x) = x$.

Example 3. Show that $|-6 + 3| = 3$.

Solution: $|-6 + 3| = |-3| = 3$.

Note how the concept of absolute value simplifies the statement of the next theorem.

THEOREM 8-1. If P and Q are points on the same vertical line, then $PQ = |y_P - y_Q|$.

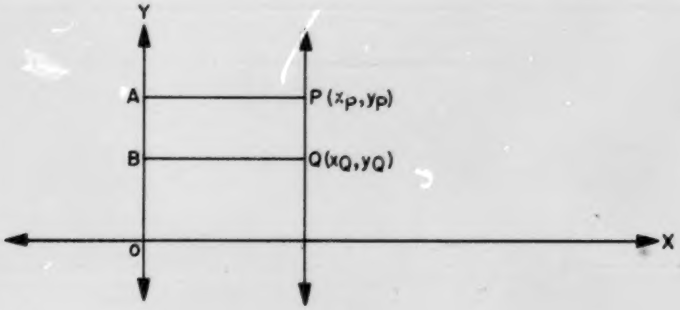
Proof:

1. If P and Q are on the y-axis, then the theorem is proved by the use of the Ruler Postulate.

2. If P and Q are not on the y-axis, let A and B be the respective projections of P and Q into the y-axis.

22

Then by our definition of y-coordinates we know that $y_P = y_A$ and $y_Q = y_B$. Now y_A and y_B are the same as the y-coordinates of A and B and therefore $AB = |y_A - y_B|$. Since ABQP is a parallelogram, it follows that AB = PQ, and hence that $PQ = |y_P - y_Q|$.



THEOREM 8-2. If P and Q are points on the same horizontal line, then $PQ = |x_P - x_Q|$.

Proof: A proof similar to that for Theorem 8-1 can be given.

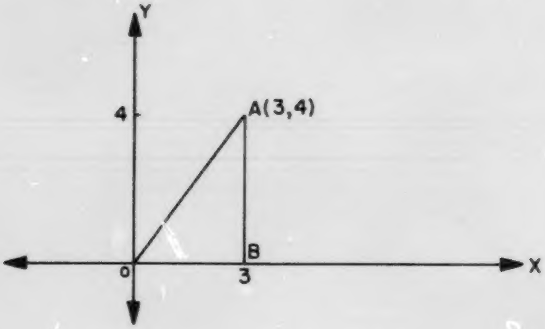
THEOREM 8-3. Every vertical line is perpendicular to every horizontal line.

Proof: This is a special case of Corollary 6-5-2.

We have seen how to use xy-coordinates to measure the distance between two points when those points are on horizontal or vertical lines. We now proceed to develop a method for finding the distance between two points that are on an oblique line. We introduce the method by means of two examples.

Example 1. Find OA if $O = (0,0)$ and $A = (3,4)$.

Solution:



Let B be the projection of A into the x-axis. Then $OB = 3$ and $BA = 4$. But ΔOBA is a right triangle. Using the Pythagorean Theorem we get

$$(OA)^2 = (OB)^2 + (BA)^2$$

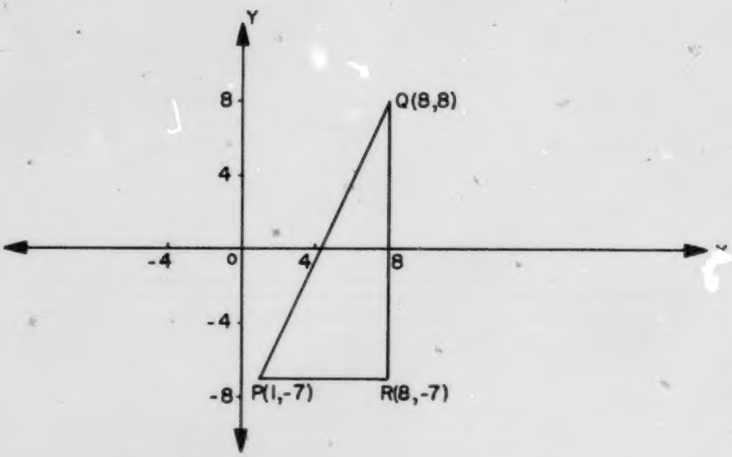
$$(OA)^2 = 9 + 16$$

$$(OA)^2 = 25$$

$$OA = 5 .$$

Example 2. Find PQ if $P = (1,-7)$ and $Q = (8,8)$.

Solution:


Y
8
Q(8,8)
4
-4
0
4
8
-4
-8
P(1,-7)
R(8,-7)

Let R be the point (8,-7) . Then PRQ is a right triangle. $PR = |8 - 1| = 7$; $QR = |8 - (-7)| = 15$; and

$$(PQ)^2 = (PR)^2 + (QR)^2 = 7^2 + 15^2 = 49 + 225 = 274 ,$$

$$PQ = \sqrt{274} .$$

We proceed to the theorem, which, once proved, will enable us to find the distance between any two points without reference to a right triangle. The result of this theorem is often referred to as the distance formula for points in a plane.

THEOREM 8-4. If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are two points in the xy-plane, then

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$

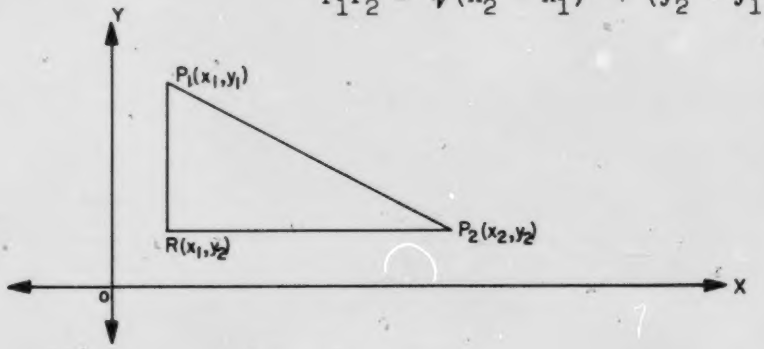
Proof: Let $R = (x_1,y_2)$. If $\overline{P_1P_2}$ is an oblique segment, then P_1RP_2 is a right triangle. Then

$(P_1P_2)^2 = (P_1R)^2 + (P_2R)^2$, $P_1R = |y_2 - y_1|$, $P_2R = |x_2 - x_1|$ and since

$$(|x_2 - x_1|)^2 = (x_2 - x_1)^2 ,$$

$$(|y_2 - y_1|)^2 = (y_2 - y_1)^2 ,$$

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$



If $\overline{P_1P_2}$ is horizontal, then $y_1 = y_2$, $R = P_1$, and $y_2 - y_1 = 0$. If $\overline{P_1P_2}$ is vertical, then $x_1 = x_2$, $R = P_2$, and $x_2 - x_1 = 0$.

In either case the relationships

$$(1)\quad (P_1P_2)^2 = (P_1R)^2 + (P_2R)^2 \text{ and}$$

$$(2)\quad P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

are still valid.

Example 1. Find AB if $A = (7,15)$ and $B = (7,13)$.

Solution: $AB = \sqrt{(7 - 7)^2 + (13 - 15)^2} = \sqrt{(-2)^2} = 2$.

Example 2. Find CD if $C = (-1,5)$ and $D = (5,-1)$.

Solution:

$$CD = \sqrt{(5 - (-1))^2 + (-1 - 5)^2} = \sqrt{36 + 36} = \sqrt{2 \times 36} = 6\sqrt{2}.$$

Example 3. The vertices of $\triangle ABC$ are $A(-1,-2)$, $B(4,0)$, $C(2,5)$. Prove that $\triangle ABC$ is a right isosceles triangle.

Proof: We have to prove (1) $\triangle ABC$ is isosceles.
(2) $\triangle ABC$ is a right triangle.

We can prove both if we know AB, BC, CA.

$$AB = \sqrt{(4 - (-1))^2 + (0 - (-2))^2} = \sqrt{5^2 + 2^2} = \sqrt{29}.$$

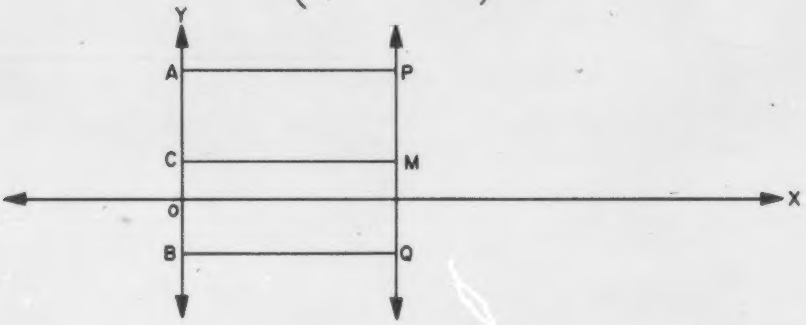
$$BC = \sqrt{(2 - 4)^2 + (5 - 0)^2} = \sqrt{(-2)^2 + 5^2} = \sqrt{29}.$$

$$CA = \sqrt{(2 - (-1))^2 + (5 - (-2))^2} = \sqrt{3^2 + 7^2} = \sqrt{58}.$$

We can see that $AB = BC$ and therefore $\triangle ABC$ is isosceles. We can see that $(AB)^2 + (BC)^2 = (CA)^2$ and therefore $\triangle ABC$ is a right triangle.

THEOREM 8-5. If P and Q are two points in the same vertical line, then the midpoint M of $\overline{PQ}$ is the point

$$M = \left(x_P \, , \, \frac{y_P + y_Q}{2}\right) .$$



Proof: Since P, Q, and M lie in the same vertical line $x_P = x_Q = x_M$. Let A, B, and C be the respective projections of P, Q, and M into the y-axis. Then $y_P = y_A$, $y_Q = y_B$ and $y_M = y_C$. Since M is the midpoint of $\overline{PQ}$ it follows from Theorem 7-2 that C is the midpoint of $\overline{AB}$. It then follows from the definition of a midpoint that $y_C = \frac{y_A + y_B}{2}$. Therefore $y_M = \frac{y_P + y_Q}{2}$ and

$$M = \left(x_P \, , \, \frac{y_P + y_Q}{2}\right) .$$

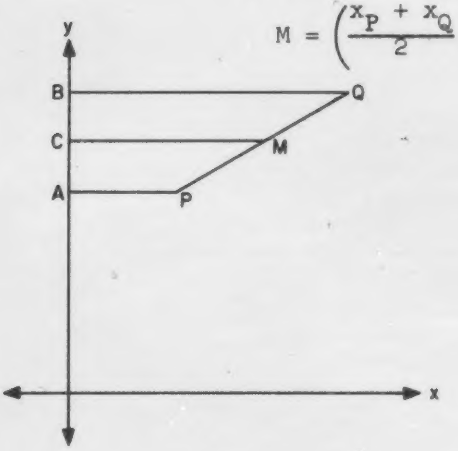
THEOREM 8-6. If P and Q are two points on the same horizontal line, then the midpoint M of $\overline{PQ}$ is the point

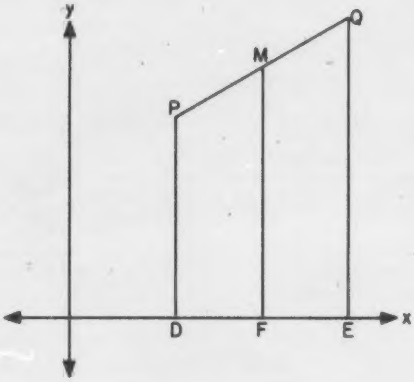
$$M = \left(\frac{x_P + x_Q}{2} \, , \, y_P\right) .$$

Proof: A proof similar to that for Theorem 8-5 may be given.

THEOREM 8-7. If P and Q are distinct points on a line which is neither vertical nor horizontal, then the midpoint M of $\overline{PQ}$ is the point

$$M = \left(\frac{x_P + x_Q}{2}, \frac{y_P + y_Q}{2}\right).$$





Proof: Let A, B, C be the respective projections of P, Q, M into the y-axis. Let D, E, F be the respective projections of P, Q, M into the x-axis. It follows from Theorem 7-2 that C is the midpoint of $\overline{AB}$ and F is the midpoint of $\overline{DE}$. It follows from the definition of a midpoint that

$$y_C = \frac{y_A + y_B}{2} \quad \text{and} \quad x_F = \frac{x_D + x_E}{2}.$$

Since $y_A = y_P$ and $y_B = y_Q$ then $y_M = \dfrac{y_P + y_Q}{2}$.

Since $x_D = x_P$ and $x_E = x_Q$ then $x_M = \dfrac{x_P + x_Q}{2}$.

Therefore $M = \left(\dfrac{x_P + x_Q}{2}, \dfrac{y_P + y_Q}{2}\right)$.

Since in Theorem 8-5 $x_P = \dfrac{x_P + x_Q}{2}$ when $x_P = x_Q$ and in Theorem 8-6 $y_P = \dfrac{y_P + y_Q}{2}$ when $y_P = y_Q$, we may combine the results of the three preceding theorems as follows:

THEOREM 8-8. If $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ are any two distinct points in a plane, then the midpoint M of $\overline{PQ}$ is the point

$$M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right) .$$

Example 1. If $P = (3,8)$ and $Q = (7,4)$, find the midpoint of $\overline{PQ}$.

Solution: Let M be the midpoint of $\overline{PQ}$.

$$M = \left(\frac{7 + 3}{2}, \frac{4 + 8}{2}\right) = (5,6) .$$

Example 2. If $R = (2,5)$ and $S = (-5,-3)$, find the midpoint M of $\overline{RS}$.

Solution:

$$M = \left(\frac{-5 + 2}{2}, \frac{-3 + 5}{2}\right) = \left(\frac{-3}{2}, \frac{2}{2}\right) = \left(\frac{-3}{2}, 1\right) .$$

Example 3. If $A = (0,0)$, $B = (0,6)$ and $C = (8,0)$, find the length of the median from A to $\overline{BC}$.

Solution: Let D be the midpoint of $\overline{BC}$. Then

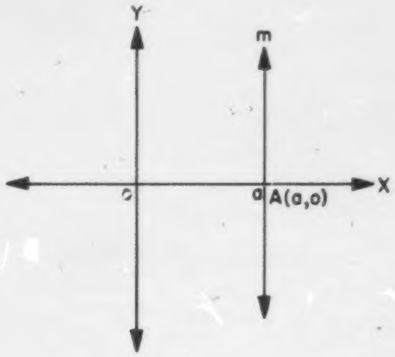
$$D = \left(\frac{8 + 0}{2}, \frac{0 + 6}{2}\right) = (4,3) .$$

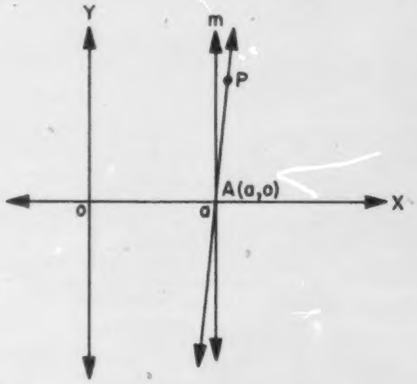
$$AD = \sqrt{(3 - 0)^2 + (4 - 0)^2} = \sqrt{3^2 + 4^2} ,$$

$$AD = \sqrt{25} = 5 .$$

29

THEOREM 8-9. Let a be any real number. Then the set of all points in the xy-plane each of which has x-coordinate a is a vertical line.





Proof: Let m be the vertical line which cuts the x-axis at the point A whose x-coordinate is a . We must establish two statements:

1. If a point lies in m , then its x-coordinate is a .
2. If a point has x-coordinate a , then it lies in m .

The first of these statements follows immediately from the definition of x-coordinate. The second statement is proved indirectly. Suppose, contrary to Statement (2), that there is a point P whose x-coordinate is a and which is not in m . Then the vertical line through P contains A . Then we have two vertical lines containing A : m and $\overleftrightarrow{PA}$. But this is impossible. Why? Therefore every point with x-coordinate a lies in m .

THEOREM 8-10. Let b be any real number. The set of all points in the xy-plane with y-coordinate b is a horizontal line.

Proof: A proof similar to that for Theorem 8-9 can be given.

Problem Set 8-3

1. Use the distance formula to find the distance between

   (a) (0,0) and (6,10) .
   (b) (0,0) and (-6,10) .
   (c) (1,2) and (6,14) .
   (d) (8,11) and (15,35) .
   (e) (3,8) and (-5,-7) .
   (f) (-2,3) and (-1,4) .
   (g) (10,1) and (49,81) .
   (h) (-6,3) and (4,-2) .
   (i) $(-3\frac{1}{2},4)$ and $(-1\frac{1}{2},0)$.
   (j) (0,3) and (-4,0) .
   (k) (8.1,6) and (5.9,4.9) .
   (l) $(3\pi,\pi)$ and $(-2\pi,-\pi)$.

2. Find the midpoint of $\overline{AB}$ if A and B have the respective coordinates

   (a) (0,0) and (6,10) .
   (b) (0,0) and (-6,10) .
   (c) (1,2) and (6,14) .
   (d) (-2,3) and (1,4) .
   (e) (-5,-2) and (-5,6) .
   (f) (a,7) and (3a,-3) .
   (g) (r,s) and (-3r,5s) .

3. (a) Write a formula for the square of the distance between (x_1,y_1) and (x_2,y_2) .

   (b) Write the following statement as an equation: The square of the distance between (0,0) and (x,y) is 25 .

4. Show that the triangles with vertices as given are right triangles. Use the distance formula to find the lengths of the sides of each triangle.

   (a) (0,0) , (3,4) , (3,0) .
   (b) (-6,2) , (5,-1) , (4,4) .
   (c) (-2,-3) , (4,5) , (-4,1) .
   (d) (1,3) , (5,6) , (4,-1) .
   (e) (13,-1) , (-9,3) , (-3,-9) .
   (f) (-4,0) , (0,6) , (9,0) .

5. The vertices of a quadrilateral are A(0,0) , B(5,0) , C(5,4) and D(0,4) .

   (a) Show that AC = BD .
   (b) Show that the midpoint of $\overline{AC}$ and the midpoint of $\overline{BD}$ is the same point.

6. The vertices of a triangle are $A(-2,1)$, $B(0,5)$ and $C(2,-1)$. Find the midpoint of $\overline{BC}$. Find the length of the median to $\overline{BC}$.

7. The vertices of a triangle are $R(-4,-2)$, $S(5,10)$ and $T(4,-2)$. Find the lengths of the medians to $\overline{ST}$ and $\overline{RS}$.

8. Find the coordinates of the midpoint C of $\overline{AB}$ if $A = (-1,0)$ and $B = (7,4)$. Then use the distance formula to verify that $AC = CB = \frac{1}{2}AB$.

9. (a) Show that $A(6,11)$ is equally distant from $B(-1,2)$ and $C(3,0)$.
   (b) Show that two of the medians in $\triangle ABC$ are equal in length.

10. Use the distance formula to show that $A(0,2)$, $B(4,8)$ and $C(6,11)$ are collinear. (Hint: Show that $AB + BC = AC$.)

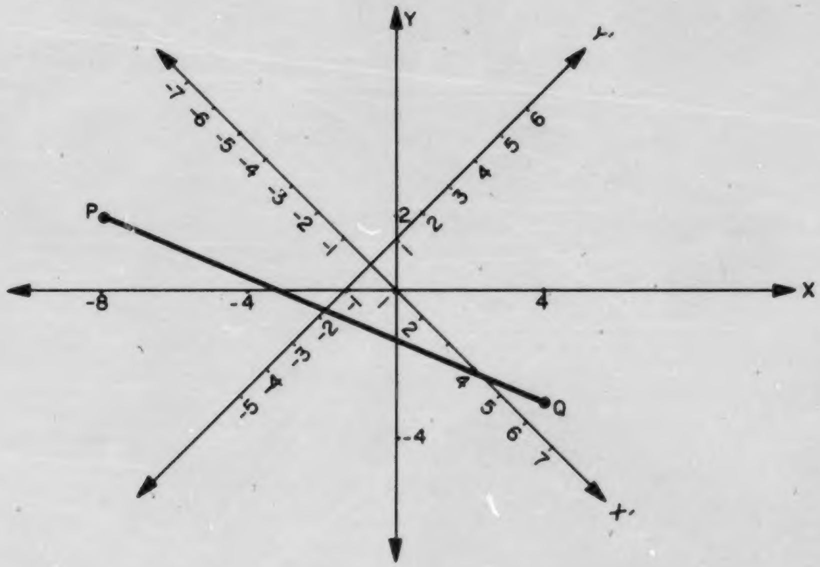
11. If the distance between $E(6,-2)$ and $F(0,y)$ is 10, find the possible y-coordinates of F.

12. Find the coordinates of the points on the x-axis whose distance from $(1,6)$ is 10.

13. Using the distance formulas, prove that $AD = BC$ if $A = (0,0)$, $D = (b,c)$, $B = (a,0)$, and $C = (a + b, c)$.

14. The vertices of a square RSTP are $R(a,a)$, $S(-a,a)$, $T(-a,-a)$, $P(a,-a)$. Show that its diagonals are congruent.

15. There are two coordinate systems in this diagram. One has axes labeled X and Y . The other has axes labeled X' and Y' . All axes have the same scale. Estimate the coordinates of P and of Q in the xy-system and then calculate the length of $\overline{PQ}$. Then estimate the coordinates of P and Q in the x'y'-system and again calculate the length of $\overline{PQ}$. Do you think that the length of $\overline{PQ}$ is independent of the choice of axes?



8-4. The Set-Builder Notation.

In our discussion of sets in Chapter 2 we considered the set of all positive integers. The underlined phrase clearly defines a certain set of numbers. In general, a set is defined by a list of its elements or by a property of its elements. If a set has an infinite number of elements, we cannot list all of its members so we use a property or properties of its elements to define it.

Consider the following property of a real number: between 3 and 5 . Some real numbers which have this property are 3.5 , 4 , 4.5 , 3.1 , 4.9 , 3.001 , and 4.999 .

Some real numbers which do not have this property are 3, 5, -7, 46, 0, -4.3, 5.7, 5.001, and 2. The set of all real numbers between 3 and 5 is a clearly defined set. A symbol which denotes this set is $\{x : 3 < x < 5\}$. We read it: the set of all x such that x is between 3 and 5. There are three parts within the braces: the set of all of something, before the colon; the colon, which is read "such that;" and a stated property after the colon.

Consider next the following property of a point (x,y): _its x-coordinate is 3, and its y-coordinate is a real number_. Some points in this set are $(3,5)$, $(3,-137)$, $(3,0)$, $(3,105732.4)$, and $(3,3)$. Some points not in this set are $(5,3)$, $(4,0)$, $(-7,2)$. A symbol which denotes this set is $\{(x,y) : x = 3\}$. We read it: the set of all points (x,y) such that $x = 3$. Frequently, as in this example, we understand that x and y are real numbers even if it is not indicated in the symbol.

In general, the symbol $\{a : \text{property}\}$, which we call the _set-builder_ symbol or notation, denotes the set of all elements a each of which has the stated property.

Example 1. Use a set-builder symbol to denote the set of all points in the first quadrant.

_Solution:* $\{P : P \text{ is a point in Quadrant I}\}$.

Alternate solution: $\{(x,y) : x > 0 \text{ and } y > 0\}$.

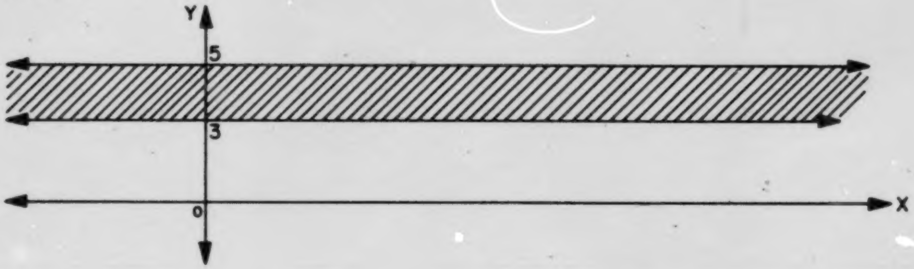
Example 2. Use a set-builder symbol to denote the set of all points whose x-coordinate is 7 and whose y-coordinate is a number greater than 5.

Solution: $\{(x,y) : x = 7 \text{ and } y > 5\}$.

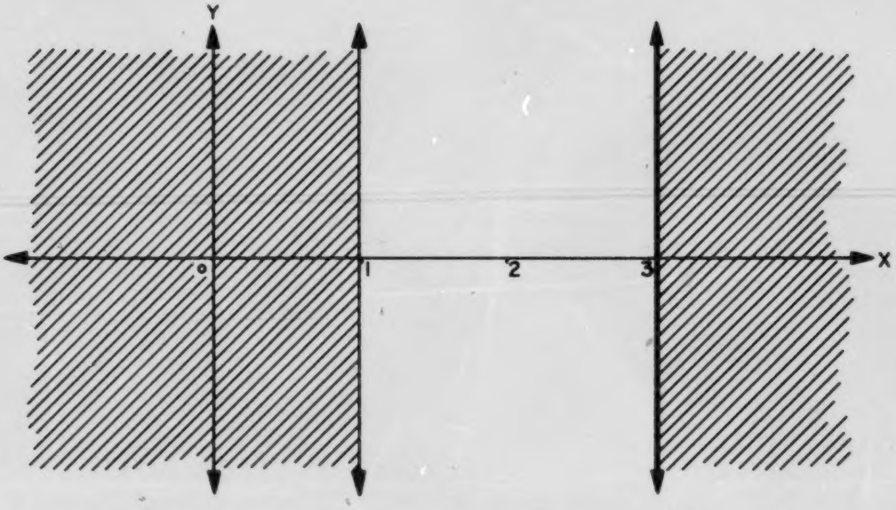
Alternate solution: $\{(7,y) : y > 5\}$.

34

Example 3. Plot the graph of the set $\{(x,y) : 3 \leq y \leq 5\}$. The set includes all points in the infinite strip between the two horizontal lines and on these lines.



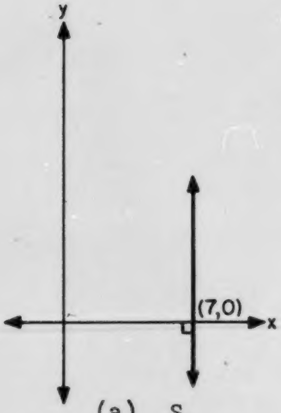
Example 4. Plot the set $\{(x,y) : x < 1 \text{ or } x \geq 3\}$. The set contains all points in the two halfplanes which are suggested by shading, and all points in the edge of one of these halfplanes.



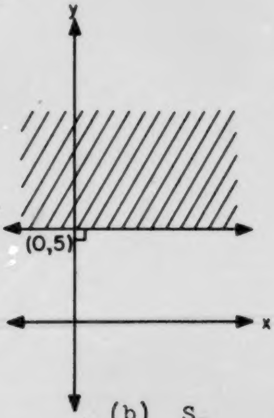
8-5. Composite Conditions.

In Example 2 above, the set $\{(x,y) : x = 7 \text{ and } y > 5\}$ was discussed and in Example 4, the set $\{(x,y) : x < 1 \text{ or } x \geq 3\}$. In the set-builder notation used to indicate these sets we note that the property listed in each case is a composite condition, that is to say, a combination of conditions. In Example 2 the conditions $x = 7$, $y \geq 5$ are connected by the word "and"; in Example 4 the conditions $x < 1$, $x \geq 3$ are connected by the word "or."

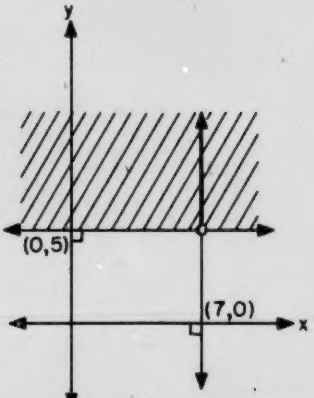
Our purpose is to illustrate briefly how these composite conditions should be interpreted in so far as they are used in our work. Let c_1 be the statement $x = 7$ and c_2 the statement $y > 5$. Let $S_1 = \{(x,y) : c_1\}$, $S_2 = \{(x,y) : c_2\}$ and $S_3 = \{(x,y) : c_1 \underline{\text{ and }} c_2\}$.



(a) S_1

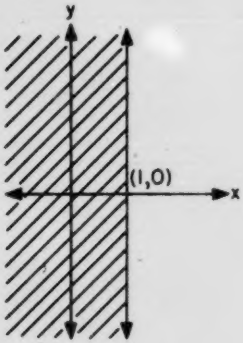


(b) S_2

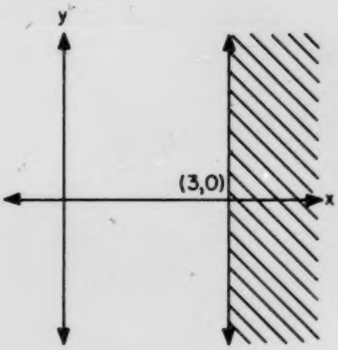


(c) S_3

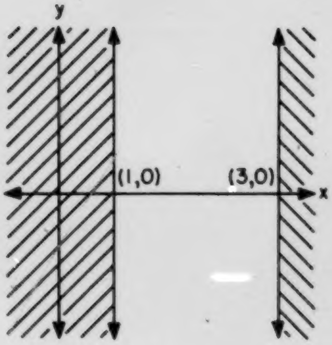
As shown in the diagrams above S_1 is a vertical line and S_2 is a halfplane. Do you see the special relationship which S_3 has to S_1 and S_2? S_3, the interior of a ray, is the intersection of S_1 and S_2. Now let c_3 be the statement $x < 1$ and c_4 the statement $x > 3$. Let $T_1 = \{(x,y) : c_3\}$, $T_2 = \{(x,y) : c_4\}$ and $T_3 = \{(x,y) : c_3 \underline{\text{ or }} c_4\}$.



(a) T_1



(b) T_2



(c) T_3

Describe T_1 ; T_2 . Now describe T_3 . Notice that in this case T_3 is the union of T_1 and T_2 .

In general when working with the set-builder notation you should remember the following:

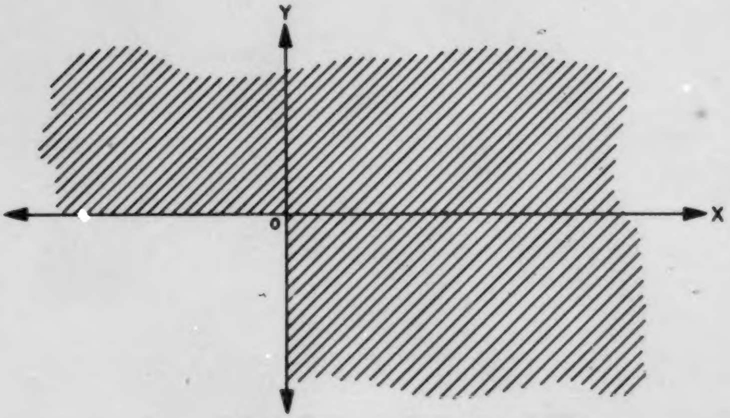
(1) A set whose defining property is a composite condition using the connective "and" can be considered the intersection of the sets determined by the individual conditions of the composite.

(2) A set whose defining property is a composite condition using the connective "or" can be considered the union of the sets determined by the individual conditions of the composite.

Example 1. What are the points in the set

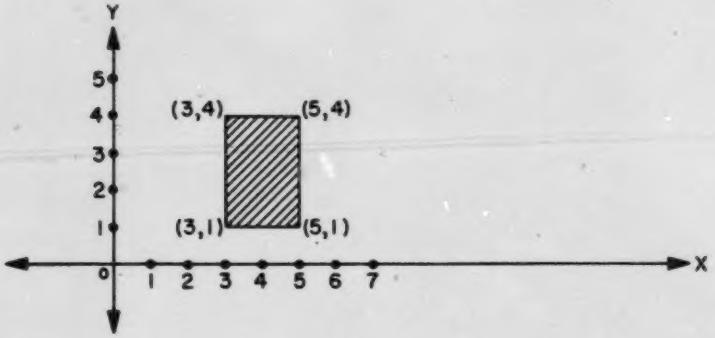
$$\{(x,y) : x > 0 \text{ or } y > 0\} \text{ ?}$$

This is the set of all points (x,y) such that $x > 0$, or $y > 0$, or both $x > 0$, $y > 0$. The set contains all points in the plane except those in the third quadrant, in the negative x-axis, in the negative y-axis, and the origin. Note that the graph is the union of the graph of $x > 0$ and the graph of $y > 0$.



Example 2. $\{(x,y) : 3 \le x \le 5 \text{ and } 1 \le y \le 4\}$.

This is the set of all points inside and on the rectangle whose vertices are $(3,1)$, $(3,4)$, $(5,4)$, and $(5,1)$.



When dealing with sets defined by a composite condition using the connective "and," a comma often is used in place of the word "and." Thus the set $\{(x,y) : x = 5 , y > 3\}$ is understood to be the same as the set $\{(x,y) : x = 5 \text{ and } y > 3\}$.

Problem Set 8-5

1. Identify the quadrant with the proper Roman numeral.

   (a) $\{(x,y) : x > 0 \text{ and } y < 0\}$ is Quadrant _____ .
   (b) $\{(x,y) : x < 0 \text{ and } y < 0\}$ is Quadrant _____ .
   (c) $\{(x,y) : x < 0 \text{ and } y > 0\}$ is Quadrant _____ .
   (d) $\{(x,y) : x > 0 \text{ and } y > 0\}$ is Quadrant _____ .
   (e) $\{(x,y) : y > 0 \text{ and } x < 0\}$ is Quadrant _____ .
   (f) $\{(x,y) : xy > 0\}$ is the union of Quadrants _____ and _____ .
   (g) $\{(x,y) : xy < 0\}$ is the union of Quadrants _____ and _____ .
   (h) $\{(x,y) : xy \neq 0\}$ is the union of _____ .
   (i) $\{(x,y) : x > 0 \text{ and } y \neq 0\}$ is the union of _____ .
   (j) $\{(x,y) : \frac{1}{x} > 0 \text{ and } \frac{1}{y} < 0\}$ is Quadrant _____ .
   (k) $\{(x,y) : x < 0 \text{ and } y = |x|\}$ is Quadrant _____ .

2. Find the coordinates of the endpoints of all possible segments which satisfy the given conditions.

   (a) $\overline{AB}$ lies on the y-axis with the origin at its midpoint; $AB = 7$.
   (b) $\overline{AB}$ is a subset of the x-axis; A, O, B are collinear in that order and $AO = \frac{1}{2}OB$; $AB = 12$.
   (c) $\overline{AB}$ is either horizontal or vertical; A is at the origin; $AB = r$.
   (d) $\overline{AB} \parallel$ to the x-axis and $\overline{AB}$ is 5 units above the x-axis; the y-axis bisects $\overline{AB}$; $AB = 8$.
   (e) $AB = 5$; A lies on the x-axis; B lies on the y-axis; $OA = OB$.
   (f) $\overline{AB}$ is in the y-axis; A is at the origin; $AB = 6$; $\overline{CD} \parallel \overline{AB}$; $CD = AB$; C is 2 units above A and 3 units to the right of A .

3. Find the coordinates of the vertices of the indicated polygons:

   (a) A coordinate system places isosceles triangle ABC so that the origin is the midpoint of base $\overline{AB}$, $\overline{AB}$ is a subset of the x-axis, C lies above the x-axis, $AB = 6$, $OC = 4$.

(b) An isosceles triangle ABC whose altitude is 3 and whose base has length 5 . The base is a subset of the y-axis, and the opposite vertex, C , is on the positive x-axis.

(c) An isosceles triangle ABC has AB = 6 , AC = BC = 5 . The origin is at the midpoint of the base, the x-axis contains the base, and C is above the x-axis.

(d) A parallelogram ABCD for which AB = 7 , A and D have coordinates (0,0) and (3,5) respectively, and $\overline{AB}$ is on the x-axis.

4. In each of the following find the coordinates of the vertices of the polygon:

(a) A right triangle ABC has $\angle C$ a right angle, CA = 21 and CB = 10 . A coordinate system places C at the origin and B in the negative x-axis.

(b) An isosceles triangle ABC has base $\overline{AB}$ of length 4 and altitude to $\overline{AB}$ of length 3 . A coordinate system places A at the origin and B in the positive x-axis.

(c) Same as (b), except that C is at the origin. A is in Quadrant I, and $\overline{AB}$ is perpendicular to the x-axis.

(d) An equilateral triangle ABC has side of length 10 . A coordinate system is established with the x-axis containing $\overline{AB}$ and the positive y-axis containing C .

5. Find the coordinates of the vertices of the polygon determined in each of the following:

(a) A right triangle ABC has $\angle C$ a right angle, CA = a and CB = b . A coordinate system places C at the origin, B in the negative x-axis, and A in the positive y-axis.

(b) An isosceles triangle ABC has base $\overline{AB}$ of length b , and altitude of length a . A coordinate system places A at the origin, B in the positive x-axis and C above the x-axis.

(c) The triangle in (c) is placed so that C is at the origin and the altitude lies along the positive x-axis.

(d) An equilateral triangle has side of length s and a coordinate system is established so that one side lies along the x-axis and the opposite vertex is in the positive y-axis.

8-6. Equations and Inequalities.

$\{(x,y) : x = 3\}$ is a line. It contains all those points and only those points in the xy-plane whose x-coordinate is 3 and whose y-coordinate is any real number. We say that $x = 3$ is an equation of the line or think of $x = 3$ as a condition imposed upon (x,y) . The condition $x = 3$ places a restriction on the x-coordinate but no restriction on the y-coordinate. Thus the line $\{(x,y) : x = 3\}$ contains infinitely many points, such as $(3,-173.447)$, $(3,-2)$, $(3,-1)$, $(3,0)$, $(3,25)$, $(3,127.3)$. Of course there are infinitely many points not on this line, such as $(2,3)$, $(2.999,-7)$, $(0,-3)$.

$\{(x,y) : x > 3\}$ is a halfplane. We say that $x > 3$ is an inequality for the halfplane. This halfplane contains all those points and only those points in the xy-plane whose x-coordinate is a real number greater than 3 and whose y-coordinate is a real number. Is $(5,-5)$ in this halfplane? Is $(-5,5)$ in this halfplane? Is $(3,3)$ in this halfplane? What set-builder notation could you use for the edge of this halfplane?

In some textbooks the set $\{(x,y) : x = 3\}$ is called the locus of the equation $x = 3$; $\{(x,y) : x > 3\}$ is called the locus of the inequality $x > 3$. In general a locus is a set determined by a condition or a combination of conditions.

Thus the locus of all points in the xy-plane which are equidistant from the lines $\{(x,y) : y = -7\}$ and $\{(x,y) : y = 13\}$ is the line $\{(x,y) : y = 3\}$. In our text however we will use the expression "the set of all points such that" rather than the term locus.

Consider the sets S and T given as follows: $S = \{(x,y) : x = 5\}$ and $T = \{(x,y) : x + 1 = 6\}$. A point (x,y) satisfies the condition $x = 5$ if and only if it satisfies the condition $x + 1 = 6$. In other words, if (a,b) is a point in T, it is also a point in S, and conversely. Hence we may write $S = T$. The equation $S = T$ is an equation involving sets (of points in the xy-plane) and you should recall that two sets are equal if and only if they have exactly the same members. This occurs if the sets are defined by equivalent equations. The equations $x = 5$ and $x + 1 = 6$ are examples of equivalent equations. In algebra you learned how to derive equivalent equations in the process of solving equations. Thus

$$\begin{aligned} 2x + 3 &= 4x + 13 \\ 3 &= 2x + 13 \\ -10 &= 2x \\ 2x &= -10 \\ x &= -5 \end{aligned}$$

are five equivalent equations. Each of them becomes a true sentence if x is replaced by -5; each of them becomes a false sentence if x is replaced by any number different from -5.

Problem Set 8-6

1. Find the distance between lines $\{(x,y) : x = 4\}$ and $\{(x,y) : x = -4\}$.
2. Write the coordinates of three points on the line $\{(x,y) : y = 3\}$.

3. Plot the set

   (a) $\{(x,y) : x = 2\}$.
   (b) $\{(x,y) : y = 3\}$.

4. Describe the union of the two sets in Problem 3. May this be written as the set $\{(x,y): x = 2$ or $y = 3\}$?

5. What is the intersection of the two sets in Problem 3? May this be written as the set $\{(x,y) : x = 2$ and $y = 3\}$? As the set $\{(x,y) : x = 2 , y = 3\}$?

6. (a) Plot the set $\{(x,y) : x = 2 \text{ and } 0 \leq y \leq 3\}$.
   (b) What geometric object does this set form?
   (c) How many elements does it contain?

7. Plot the set of points whose coordinates are given below and describe the graph in each case.

   (a) $\{x : x \leq 3\}$.
   (b) $\{(x,y) : x < 3\}$.
   (c) $\{y : y \leq 2 \text{ or } y \geq 4\}$.
   (d) $\{(x,y) : y < 2 \text{ or } y > 4\}$.

8. Plot and describe each of the graphs given below:

   (a) The union of $\{(x,y) : x > 3\}$ and $\{(x,y) : y \leq 3\}$. Express this union with one set-builder symbol.
   (b) The intersection of $\{(x,y) : x \leq 2\}$ and $\{(x,y) : y \geq -2\}$. Express this set with one set-builder symbol.
   (c) The set $\{(x,y) : x \geq 0 \text{ and } y \leq 3\}$.
   (d) The set $\{(x,y) : -4 < x \leq 2 \text{ and } -2 \leq y \leq 5\}$.

9. Express in set-builder notation the set of all points in the xy-plane which satisfy the following conditions:

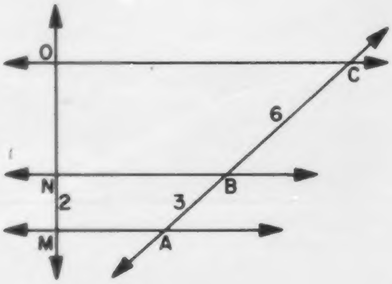
   (a) A set of points at a distance of 5 from the line whose equation is $y = 2$.
   (b) A set of points 4 units from the y-axis.
   (c) A set of points 3 units above the x-axis and 5 units to the left of the y-axis.
   (d) A set of points the same distance from the point $A(0,3)$ as from the point $B(0,-3)$. If P is any point in this set, prove $PA = PB$. If P is any point such that $PA = PB$, prove P is in this set.

10. Which of the following are pairs of equal sets?

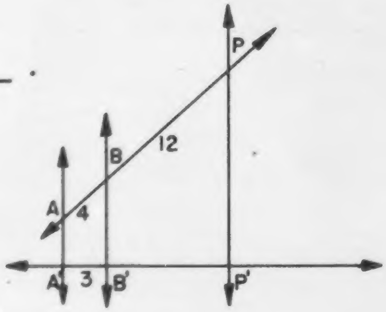
(a) $\{(x,y) : 3x + 6 = x + 8\}$ and $\{(x,y) : 2x = 2\}$.
(b) $\{(x,y) : x + 3 > 7\}$ and $\{(x,y) : x > 4\}$.
(c) $\{(x,y) : 5x - 2 < 2x + 4\}$ and $\{(x,y) : 7x < 21\}$.
(d) $\{(x,y) : -2x + 4 < 8\}$ and $\{(x,y) : x < 2\}$.
(e) $\{x : \frac{6}{x} \geq 3\}$ and $\{x : 6 \geq 3x\}$.

*11. Answer the questions indicated by filling in the blanks in each of the following:

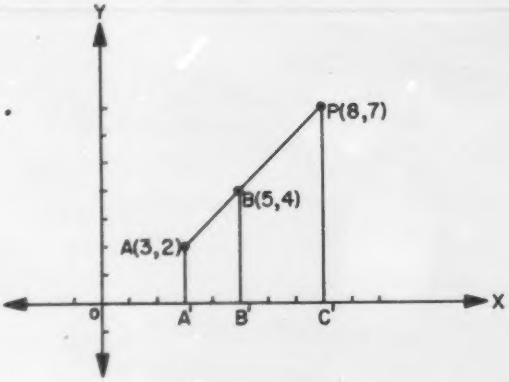
(a) If $\overleftrightarrow{OC}$, $\overleftrightarrow{NB}$, $\overleftrightarrow{MA}$ are parallel and $AB = 3$; $BC = 6$; $MN = 2$.
Then $BC = k\ AB$; $k =$ ___
$NO = t\ MN$; $t =$ ___ ;
$AC = k'\ AB$; $k' =$ ___ ;
and $MO =$ ___ MN .



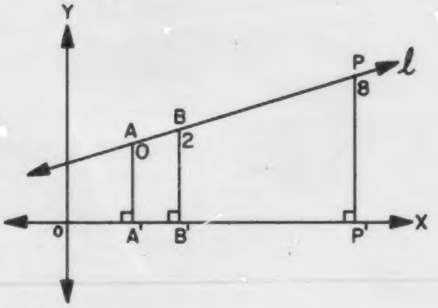
(b) If $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'} \parallel \overleftrightarrow{PP'}$,
then $AP = k\ AB$; $k =$ ___ .
$A'P' =$ ___ $A'B'$,
and $A'P' =$ ___ .



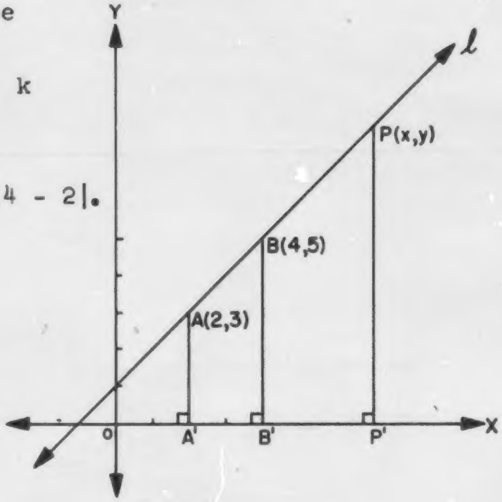
(c) $A'C' =$ ___ .
$A'B' =$ ___ .
$A'C' =$ ___ $A'B'$.
$AP =$ ___ AB .



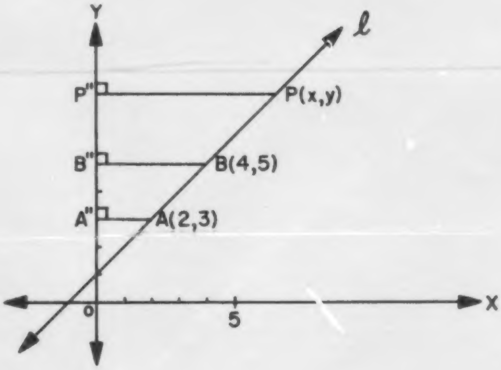
(d) If a coordinate system on ℓ assigns the coordinates 0, 2, 8 to A, B, P respectively,
A'P' = ___?___ A'B' .


Y
P
ℓ
8
B
A
2
O
0
A'
B'
P'
X

(e) The (A,B)-coordinate system on line ℓ assigns coordinate k to point P .
AP = k AB . Then
A'P' = k A'B' = k |4 - 2|.
Why?
OP' = ___?___ .
x = ___?___ .

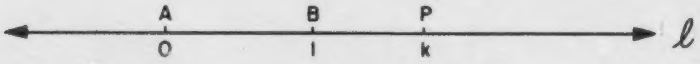

Y
ℓ
P(x,y)
B(4,5)
A(2,3)
0
A'
B'
P'
X

(f) AP = k AB .
A"P" = k |5 - 3| . Why?
OP" = ___?___ .
y = ___?___ .


Y
ℓ
P"
P(x,y)
B"
B(4,5)
A"
A(2,3)
0
5
X

(g) A coordinate system on line ℓ assigns coordinates 0, 1, k to A, B, P respectively. Where does P lie if k has values as indicated below?

(1) $k > 1$.
(2) $k = 1$.
(3) $0 < k < 1$.
(4) $k \leq 0$.
(5) $k = 0$.

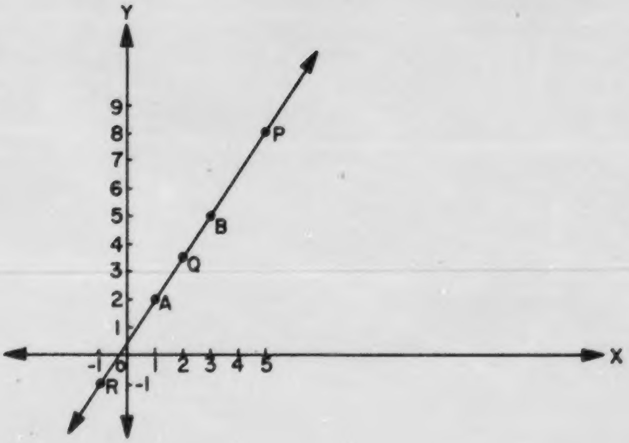


8-7. Finding the Coordinates of the Points of a Line.

We have seen that $\{(x,y) : x = 3\}$ is a vertical line. It is understood that y may be any real number.

It is natural to ask if there is an expression something like this for an oblique line. Actually there is, and it is a useful tool in geometry.

To show how to find such an expression we consider a particular line, the line $\overleftrightarrow{AB}$ where $A = (1,2)$ and $B = (3,5)$. Since the line $\overleftrightarrow{AB}$ is determined by the points A and B , it seems reasonable that we should be able to find the coordinates of other points in $\overleftrightarrow{AB}$. For example, if P is in $\overrightarrow{AB}$ and if $AP = 2AB$ we should be able to find the coordinates of P .

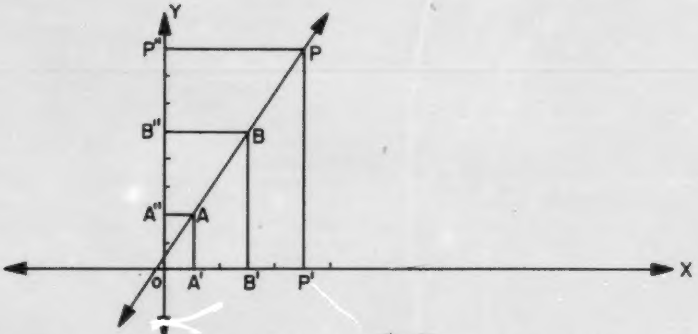


If Q is in $\overrightarrow{AB}$ and $AQ = \frac{1}{2} AB$, we should be able to find the coordinates of Q. If R is in the ray opposite to $\overrightarrow{AB}$ and if $AR = AB$, we should be able to find the coordinates of R. Actually, $P = (5,8)$, $Q = (2, 3\frac{1}{2})$, $R = (-1,-1)$. We can get these coordinates by working an individual problem for each point. But our objective here is to derive an expression from which the coordinates of P, Q, R or for that matter any other point on $\overleftrightarrow{AB}$ can be obtained by simple replacements.

In Chapter 3 we studied a coordinate system on a line. At the beginning of the present chapter we defined an xy-coordinate system in terms of two coordinate systems on lines: the x-coordinate system on the x-axis and the y-coordinate system on the y-axis. We wish now to consider a coordinate system on the line $\overleftrightarrow{AB}$. We call it the (A,B)-coordinate system. In this coordinate system, the coordinate of A is 0 and the coordinate of B is 1. For the points A, B, P, Q, R we have coordinates as tabulated.

	x-coordinate	y-coordinate	(A,B)-coordinate
A	1	2	0
B	3	5	1
P	5	8	2
Q	2	$3\frac{1}{2}$	$\frac{1}{2}$
R	-1	-1	-1
S	?	?	k

The expression which we shall derive shows us how to compute the x- and y-coordinates of a point in terms of its (A,B)-coordinates.



The (A,B)-coordinate system on $\overleftrightarrow{AB}$ established a one-to-one correspondence between the set of all real numbers k and the set of all points in $\overleftrightarrow{AB}$. If $k > 0$, the corresponding point is in $\overrightarrow{AB}$ (but not A itself); if $k = 0$, the corresponding point is A; and if $k < 0$, the corresponding point is in the ray opposite to $\overrightarrow{AB}$.

Let k be any real number and $P(x,y)$ the corresponding point in $\overleftrightarrow{AB}$. Then $AP = k\ AB$ if $k \geq 0$, $AP = -k\ AB$ if $k \leq 0$. Let P', A', B' be the respective projections of P, A, B into the x-axis. Let P'', A'', B'' be the respective projections of P, A, B into the y-axis. (If P is in the x-axis, then $P = P'$; if P is in the y-axis, then $P = P''$.) From Theorem 7-3 it follows that the segments formed by A', B', P' on the x-axis and the segments formed by A'', B'', P'', on the y-axis are proportional to the corresponding segments in the line $\overleftrightarrow{AB}$. Therefore

if $k \geq 0$	if $k \leq 0$
$AP = k\ AB$	$AP = -k\ AB$
(1) $A'P' = k\ A'B'$	(1) $A'P' = -k\ A'B'$
$\lvert x - 1\rvert = k\lvert 3 - 1\rvert$	$\lvert x - 1\rvert = -k\lvert 3 - 1\rvert$
Since $x \geq 1$	Since $x \leq 1$
$x - 1 = 2k$	$-x + 1 = -2k$
$x = 1 + 2k$	$x = 1 + 2k$
(2) $A''P'' = k\ A''B''$	(2) $A''P'' = -k\ A''B''$
$\lvert y - 2\rvert = k\lvert 5 - 2\rvert$	$\lvert y - 2\rvert = -k\lvert 5 - 2\rvert$
Since $y \geq 2$	Since $y \leq 2$
$y - 2 = 3k$	$-y + 2 = -3k$
$y = 2 + 3k$	$y = 2 + 3k$

It follows that $P = (1 + 2k, 2 + 3k)$ and that $\overleftrightarrow{AB} = \{(x,y) : x = 1 + 2k,\ y = 2 + 3k,\ k \text{ is real}\}$.

The equations $x = 1 + 2k$, $y = 2 + 3k$ are called parametric equations for the line $\overleftrightarrow{AB}$; the symbol k is called the parameter. Each value of the parameter yields exactly one point on the line, the point $(1 + 2k, 2 + 3k)$. The value of k is the (A,B)-coordinate of the point; it tells us that the point is in $\overrightarrow{AB}$ if $k \geq 0$, in the ray opposite to $\overrightarrow{AB}$ if $k \leq 0$, and that P is $|k|$ times as far from A as B is. The following table shows several values of k and their corresponding points.

k	$x = 1 + 2k$	$y = 2 + 3k$	$P(x,y)$
0	1	2	(1,2)
1	3	5	(3,5)
-1	-1	-1	(-1,-1)
2	5	8	(5,8)
7	15	23	(15,23)
$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(-\frac{1}{3},0)$
1000	2001	3002	(2001,3002)

If we think of $A(1,2)$ as (x_1,y_1) and $B(3,5)$ as (x_2,y_2), then the parametric equations for $\overleftrightarrow{AB}$ can be written as

$$x = x_1 + k(x_2 - x_1)\ ,\ y = y_1 + k(y_2 - y_1)\ .$$

Note how these formulas resemble that of Theorem 3-6.

Are these formulas true for any oblique line determined by two points (x_1,y_1) and (x_2,y_2)? Although we could prove that they are by constructing a proof similar to that for $\overleftrightarrow{AB}$ in the above illustration, we shall not write it out here. It is natural to ask whether we can write parametric equations for horizontal and vertical lines. You will find that we can in the next problem set. These results are consolidated in the following theorem.

THEOREM 8-11. If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are any two points, then
$\overleftrightarrow{P_1P_2} = \{(x,y) : x = x_1 + k(x_2 - x_1),\ y = y_1 + k(y_2 - y_1),\ k \text{ is real}\}$.

According to Theorem 8-11 every line in the xy-plane can be "represented" by a pair of parametric equations. A natural question is: Does every pair of parametric equations represent some line? The answer to this question is no. Consider, for example, the set

$$S = \{(x,y) : x = 1 + k\cdot 0,\ y = 2 + k\cdot 0,\ k \text{ is real}\}$$

It is easy to see that $x = 1$ and $y = 2$ for every value of k and hence that S is a set whose only element is the point $(1,2)$.

However there is a method of identifying those parametric equations which do represent a line in a plane. We state it as our next theorem.

THEOREM 8-12. If a, b, c, d are real numbers such that b and d are not both zero and if
$S = \{(x,y) : x = a + bk, y = c + dk, k \text{ is real}\}$,
then S is a line.

Proof: Taking $k = 0$ and $k = 1$ we get two points in S : $A(a,c)$ and $B(a + b, c + d)$. From Theorem 8-11 it follows that:

$$\overleftrightarrow{AB} = \{(x,y) : x = a + bk, y = c + dk, k \text{ is real}\};$$

therefore $\overleftrightarrow{AB} = S$ and S is a line.

Example.

If $T = \{(x,y) : x = -2 + 3k, y = 7 + 2k, k \text{ is real}\}$, then T is a line.

Proof: If $k = 0$, then $x = -2$, $y = 7$. If $k = 1$, then $x = 1$, $y = 9$. Thus $A(-2,7)$, and $B(1,9)$ are two points in T . Using Theorem 8-11 we get

$$\overleftrightarrow{AB} = \{(x,y) : x = -2 + 3k, y = 7 + 2k, k \text{ is real}\} .$$

Therefore $\overleftrightarrow{AB} = T$, and T is a line.

We can use parametric equations for a line in expressing the coordinates of the points of a line segment or a ray. If k_1, k_2, k_3 correspond to P_1, P_2, P_3 , respectively, then k_2 is between k_1 and k_3 if and only if P_2 is between P_1 and P_3 . This follows from the properties of coordinate systems on a line as discussed in Chapter 3. Thus we get segments or rays simply by restricting the values which k may have. For example, $\overline{AB}$, where $A = (1,2)$ and $B = (3,5)$ is $\{(x,y) : x = 1 + 2k, y = 2 + 3k, 0 \leq k \leq 1\}$. Similarly, $\overrightarrow{AB}$ is $\{(x,y) : x = 1 + 2k, y = 2 + 3k, k \geq 0\}$.

Example 1.

Given $A = (3,0)$, $B = (-1,2)$. Using Theorem 8-11 express, using set-builder notation,

(a) $\overleftrightarrow{AB}$,
(b) $\overline{AB}$,
(c) $\overrightarrow{AB}$,
(d) the ray opposite $\overrightarrow{AB}$; also find
(e) the midpoint of $\overline{AB}$, and
(f) the point P such that A is between P and B and $PA = AB$.

Solution: In this problem we take $x_1 = 3$, $y_1 = 0$, $x_2 = -1$, $y_2 = 2$. Thus $x_2 - x_1 = -4$, $y_2 - y_1 = 2$. Then

(a) $\overleftrightarrow{AB} = \{(x,y) : x = 3 - 4k , y = 2k , k \text{ is real}\}$.
(b) $\overline{AB} = \{(x,y) : x = 3 - 4k , y = 2k , 0 \leq k \leq 1\}$.
(c) $\overrightarrow{AB} = \{(x,y) : x = 3 - 4k , y = 2k , k \geq 0\}$.
(d) the ray opposite $\overrightarrow{AB}$ is
$\{(x,y) : x = 3 - 4k , y = 2k , k \leq 0\}$.
(e) the midpoint of $\overline{AB}$ (set $k = \frac{1}{2}$) is
$(3 - 4 \cdot \frac{1}{2} , 2 \cdot \frac{1}{2}) = (1,1)$.
(f) the point P (set $k = -1$) is $(7,-2)$.

Example 2.

Given $A = (0,4)$ and $B = (3,0)$. Find the point C on $\overleftrightarrow{AB}$ whose x-coordinate is -2 .

Solution: In this problem we take $x_1 = 0$, $y_1 = 4$, $x_2 = 3$, $y_2 = 0$. Then $x_2 - x_1 = 3$, $y_2 - y_1 = -4$ and

$$\overleftrightarrow{AB} = \{(x,y) : x = 3k , y = 4 - 4k , k \text{ is real}\} .$$

We set $x = -2$. Then $-2 = 3k$, $k = -\frac{2}{3}$, and $y = 4 - 4(-\frac{2}{3}) = 4 + \frac{8}{3} = \frac{20}{3} = 6\frac{2}{3}$. Therefore $C = (-2,6\frac{2}{3})$.

In Example 1, Part (e), we found the midpoint of a particular segment $\overline{AB}$. The midpoint of $\overline{AB}$ is obtained by setting $k = \frac{1}{2}$ in the parametric equations. In general

$$x = x_A + \frac{1}{2}(x_B - x_A) = x_A + \frac{1}{2}x_B - \frac{1}{2}x_A = \frac{1}{2}x_A + \frac{1}{2}x_B = \frac{x_A + x_B}{2} ,$$

$$y = y_A + \frac{1}{2}(y_B - y_A) = y_A + \frac{1}{2}y_B - \frac{1}{2}y_A = \frac{1}{2}y_A + \frac{1}{2}y_B = \frac{y_A + y_B}{2} ,$$

and therefore the midpoint is

$$\left(\frac{x_A + x_B}{2} , \frac{y_A + y_B}{2}\right) .$$

Notice that this result checks with the result derived in Theorem 8-8.

Problem Set 8-7

1. Using parametric equations and set-builder notation express $\overleftrightarrow{AB}$, $\overline{AB}$, $\overrightarrow{AB}$, and the ray opposite to $\overrightarrow{AB}$ if

   (a) $A = (1,4)$, $B = (2,6)$.
   (b) $A = (-1,3)$, $B = (2,0)$.
   (c) $A = (0,0)$, $B = (3,2)$.
   (d) $A = (1,1)$, $B = (4,4)$.
   (e) $A = (-1,3)$, $B = (1,-2)$.
   (f) $A = (-3,-2)$, $B = (0,1)$.
   (g) $A = (a,b)$, $B = (c,d)$, $c \neq a$.
   (h) $A = (a,2a)$, $B = (3a,4a)$, $a \neq 0$.

2. Find the coordinates of the midpoint of $\overline{AB}$ in Problem 1(a) ; 1(b) .

3. Using the midpoint formula find the coordinates of the midpoint of the segment with the given endpoints.

   (a) $(5,7)$ and $(11,17)$.
   (b) $(-9,3)$ and $(-2,-6)$.
   (c) $(\frac{1}{2},8)$ and $(\frac{2}{3},-3)$.
   (d) $(3.5,-6)$ and $(1.7,-6)$
   (e) $(a,-b)$ and $(-a,b)$.
   (f) $(r + s, r - s)$ and $(-r,s)$.

4. In each of the following the endpoint and midpoint of a segment are given in that order. Find the coordinates of the other endpoint.

   (a) $(4,0)$ and $(9,0)$.
   (b) $(3,0)$ and $(5,2)$.
   (c) $(-2,3)$ and $(3,5)$.
   (d) $(6,2)$ and $(2,-1)$.
   (e) (a,b) and $(5a,3b)$.
   (f) $(3r,s)$ and $(0,4s)$.

5. Find the coordinates of the trisection point of $\overline{AB}$ nearer A in Problem 1(c) ; 1(d) .

6. Find the coordinates of the trisection point of $\overline{AB}$ nearer B in Problem 1(e).

7. Find the coordinates of P in $\overrightarrow{AB}$ in Problem 1(b) such that

   (a) $AP = 2AB$.
   (b) $AP = 100AB$.
   (c) $AP = \sqrt{3}\ AB$.
   (d) $AP = \pi\ AB$.

8. Find the coordinates of P in the ray opposite to $\overrightarrow{AB}$ in Problem 1(e) if

   (a) $AP = 2AB$.
   (b) $AP = 20AB$.
   (c) $AP = 3.5\ AB$.
   (d) $AP = \frac{1}{2}\ AB$.

9. Find the coordinates of P in $\overleftrightarrow{AB}$ if $A = (-1,5)$ and $B = (3,-2)$, and

   (a) $AP = 3PB$.
   (b) $BP = 4PA$.
   (c) $BA = \frac{1}{2}\ BP$.
   (d) $PA = 5BA$.

10. (a) Let $C = (-1,2)$, $D = (5,2)$. Is $\overleftrightarrow{CD}$ vertical, horizontal, or oblique? Use Theorem 8-11 to express $\overleftrightarrow{CD}$. Try three different values of k to see if the three points are on $\overleftrightarrow{CD}$.

    (b) Using $C = (x_1,a)$, $D = (x_2,a)$, $x_1 \neq x_2$, show that Theorem 8-11 is true for horizontal lines.

    (c) Using $E = (a,y_1)$, $F = (a,y_2)$, $y_1 \neq y_2$, show that Theorem 8-11 is true for vertical lines.

11. Using parametric equations and set-builder notation express the sides of the triangle whose vertices are:

    (a) $A(0,0)$, $B(0,3)$, $C(4,0)$.
    (b) $D(-3,0)$, $E(0,3)$, $F(3,0)$.

12. Draw the graph:

    (a) $\{(x,y) : x = 1 + 2k , y = 2 - k , k \text{ is real}\}$.
    (b) $\{(x,y) : x = 2k , y = k , 0 \leq k \leq 2\}$.
    (c) $\{(x,y) : x = -1 + k , y = -k , k \geq 0\}$.
    (d) $\{(x,y) : x = k , y = -k , k \leq 0\}$.
    (e) $\{(x,y) : x = 3 , y = k , -2 \leq k \leq 3\}$.

13. Given $A = (-1,3)$, $B = (2,-2)$, and C is on $\overleftrightarrow{AB}$,

    (a) find the y-coordinate of C if its x-coordinate is 5 .
    (b) find the x-coordinate of C if its y-coordinate is 8 .
    (c) find the y-coordinate of C if its x-coordinate is 29 .
    (d) find the coordinate of C if it is on the x-axis.
    (e) find the coordinate of C if it is on the y-axis.

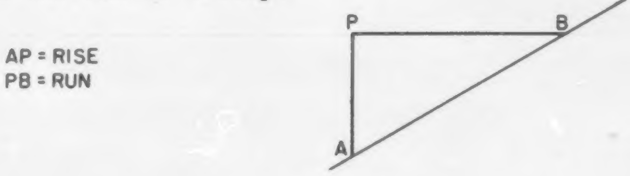
14. The vertices of a triangle are $A(0,0)$, $B(9,0)$, $C(3,6)$. Find the coordinates of D , the midpoint of $\overline{AB}$; E , the midpoint of $\overline{BC}$; and F , the midpoint of $\overline{CA}$. Show that a trisection point of each median of triangle ABC is $G(4,2)$.

15. Given $p = \{(x,y) : x = a + ck , y = b + dk , k \text{ is real}\}$.

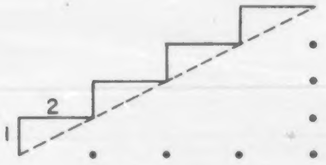
    (a) Show that p is a vertical line if $c = 0$.
    (b) Show that p is a horizontal line if $d = 0$.
    (c) Show that p contains the origin if $a = 0 = b$.

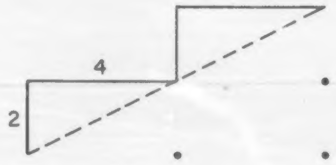
8-8. Slope.

We are now ready to study one of the important properties of a line which corresponds to the idea of the steepness of inclination of a line in the world of everyday affairs. The steepness of a stairway depends on the relationship between the rise and the run of a step.



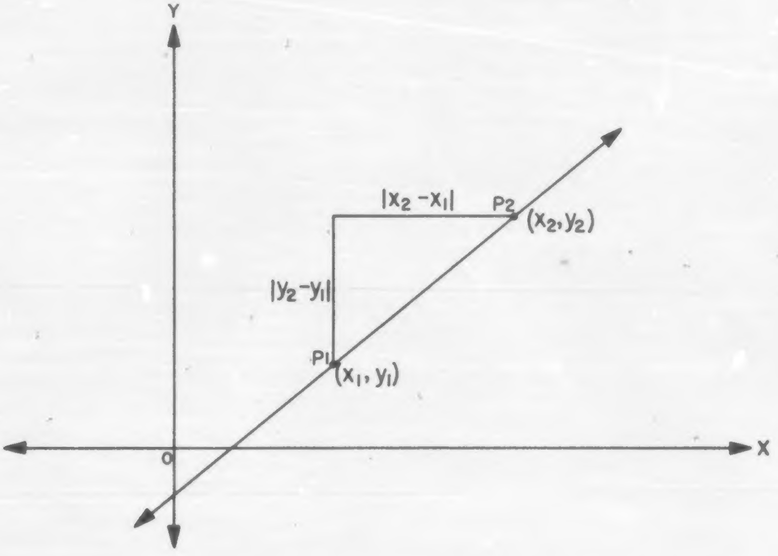
If one stairway has steps with a certain rise and run and another stairway has steps with rise and run each twice as large, is it clear that the steepness of the two stairways is the same? In other words, a run of 2 with a rise of 1 gives the same steepness as a run of 4 with a rise of 2 .



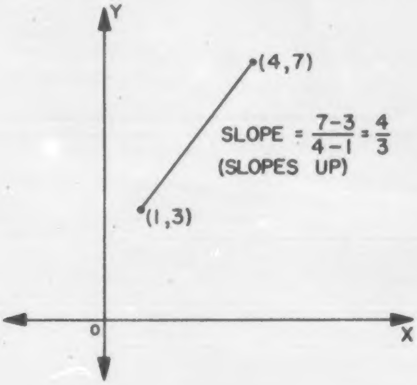


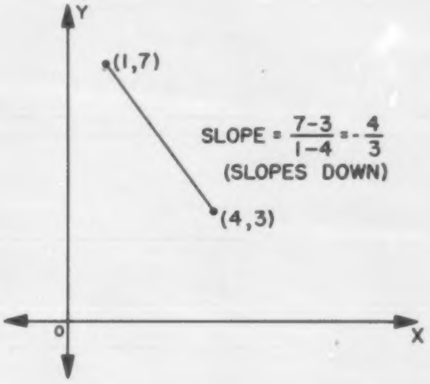
The steepness or pitch of these stairways may be defined as the number obtained by dividing the rise by the run, $\frac{1}{2}$ in either case.

The concept of the slope of a line is based on the idea of "rise divided by run." If we think of one step connecting two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ on a non-vertical line, then the rise is $|y_2 - y_1|$ and the run is $|x_2 - x_1|$.



We could define the slope of the segment $\overline{P_1P_2}$ as rise divided by run, i.e., $\frac{|y_2 - y_1|}{|x_2 - x_1|}$. But we do not. The slope of a segment $\overline{P_1P_2}$ is defined as $\frac{y_2 - y_1}{x_2 - x_1}$. The formula without the absolute value is easier to handle, and it turns out to be more useful. The absolute value of the slope conveys only the magnitude of the slope. The sign of the slope conveys the additional idea of "slopes up or down" as suggested in the figure.





Starting with the concept of the slope of a segment we now develop the concept of the slope of a line. Consider the line $\overleftrightarrow{AB}$ where $A = (1,2)$ and $B = (3,5)$. Then

$$\overleftrightarrow{AB} = \{(x,y) : x = 1 + 2k , y = 2 + 3k , k \text{ is real}\} .$$

Let us compute the slopes of several segments $\overline{P_1P_2}$ on $\overleftrightarrow{AB}$. Take P_1 as the point corresponding to k_1 and P_2 as the point corresponding to k_2 .

k_1	k_2	P_1	P_2	slope of $\overline{P_1P_2}$
0	1	(1,2)	(3,5)	$\frac{5-2}{3-1} = \frac{3}{2}$
0	3	(1,2)	(7,11)	$\frac{11-2}{7-1} = \frac{9}{6} = \frac{3}{2}$
0	-3	(1,2)	(-5,-7)	$\frac{-7-2}{-5-1} = \frac{-9}{-6} = \frac{3}{2}$
4	-4	(9,14)	(-7,-10)	$\frac{-10-14}{-7-9} = \frac{-24}{-16} = \frac{3}{2}$
k_1	k_2	$(1 + 2k_1, 2 + 3k_1)$	$(1 + 2k_2, 2 + 3k_2)$	$\frac{3}{2}$

Note that the slope of every segment of $\overleftrightarrow{AB}$ is $\frac{3}{2}$. Note also that 3 and 2 are the coefficients of k in the equations for y and x respectively. Let us check the last line of the table. Suppose k_1 and k_2 are any two distinct values of k. Substituting in the parametric equations we get

if $k = k_1$, $x_1 = 1 + 2k_1$, $y_1 = 2 + 3k_1$, $P_1 = (1 + 2k_1, 2 + 3k_1)$;

if $k = k_2$, $x_2 = 1 + 2k_2$, $y_2 = 2 + 3k_2$, $P_2 = (1 + 2k_2, 2 + 3k_2)$.

Then

$$x_2 - x_1 = (1 + 2k_2) - (1 + 2k_1) = 2(k_2 - k_1) ,$$

$$y_2 - y_1 = (2 + 3k_2) - (2 + 3k_1) = 3(k_2 - k_1) ,$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3(k_2 - k_1)}{2(k_2 - k_1)} = \frac{3}{2} .$$

Does every nonvertical line have the property that all of its segments have the same slope? We show that this is indeed the case.

Let c be any line and let $C_1(x_1,y_1)$ and $C_2(x_2,y_2)$ be any two points on c. Then

$$\overleftrightarrow{C_1C_2} = \{(x,y) : x = x_1 + k(x_2 - x_1),\ y = y_1 + k(y_2 - y_1),\ k \text{ is real}\}.$$

As in the example above we take two distinct values of k, say p and q, corresponding to two distinct points P and Q in $\overleftrightarrow{C_1C_2}$, and find

$$x_P = x_1 + p(x_2 - x_1) \quad \text{and} \quad y_P = y_1 + p(y_2 - y_1) \,,$$
$$x_Q = x_1 + q(x_2 - x_1) \quad \text{and} \quad y_Q = y_1 + q(y_2 - y_1) \,.$$

Then

$$y_P - y_Q = [y_1 + p(y_2 - y_1)] - [y_1 + q(y_2 - y_1)] = (p - q)(y_2 - y_1).$$
$$x_P - x_Q = [x_1 + p(x_2 - x_1)] - [x_1 + q(x_2 - x_1)] = (p - q)(x_2 - x_1)$$

Before we divide $y_P - y_Q$ by $x_P - x_Q$ we should assure ourselves that $x_P - x_Q \neq 0$. If $x_P - x_Q = 0$, then $x_P = x_Q$ and $\overleftrightarrow{C_1C_2}$ is a vertical line. If $\overleftrightarrow{C_1C_2}$ is a nonvertical line, $x_P - x_Q \neq 0$, $x_1 - x_2 \neq 0$, and

$$\frac{y_P - y_Q}{x_P - x_Q} = \frac{y_2 - y_1}{x_2 - x_1} \,.$$

This proves that all segments of a non-vertical line have the same slope. We may then write the following definition and theorem:

> DEFINITION. The slope of a non-vertical line is equal to the slope of any of its segments; the slope of a non-vertical ray is the slope of the line which contains the ray.
>
> Notation. The slope of $\overleftrightarrow{AB}$, $\overrightarrow{AB}$, $\overline{AB}$ is denoted by $m_{\overleftrightarrow{AB}}$, $m_{\overrightarrow{AB}}$, $m_{\overline{AB}}$, respectively.

THEOREM 8-13. The slope of a non-vertical line p is $\dfrac{y_2 - y_1}{x_2 - x_1}$, where $\overline{P_1P_2}$ is any segment of p and $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$.

Using Theorem 8-11 we can write parametric equations for a line which passes through two given points. In the following theorem we see how to write parametric equations for a line passing through a given point and having a given slope.

THEOREM 8-14. If p is the line through (x_1,y_1) with slope $m = \frac{f}{g}$, then

1. $p = \{(x,y) : x = x_1 + kg \, , \; y = y_1 + kf \, , \; k \text{ is real}\}$

   and

2. $p = \{(x,y) : x = x_1 + k \, , \; y = y_1 + km \, , \; k \text{ is real}\}$.

Proof: Suppose h is a number such that $(x_1 + g \, , \, y_1 + h)$ is a point on p . Then the slope of p is

$$\frac{(y_1 + h) - y_1}{(x_1 + g) - x_1} = \frac{h}{g} \, .$$

Since the slope is $\frac{f}{g}$ by hypothesis, it follows that $\frac{h}{g} = \frac{f}{g}$ and $h = f$. Therefore (x_1,y_1) and $(x_1 + g \, , \, y_1 + f)$ are two points on ℓ and it follows from Theorem 8-11 that

(1) $p = \{(x,y) : x = x_1 + kg \, , \; y = y_1 + kf \, , \; k \text{ is real}\}$.

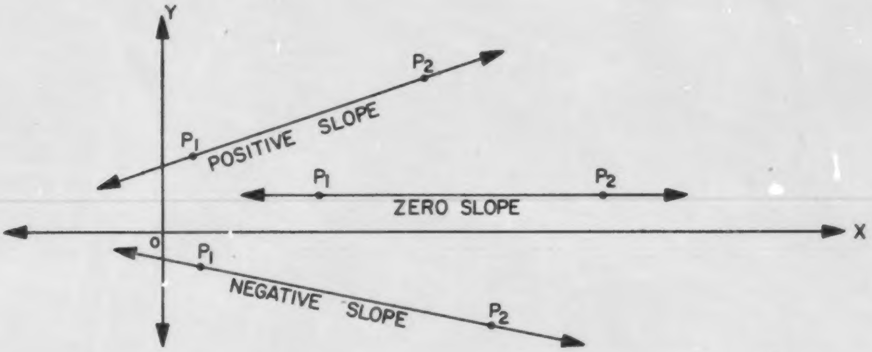
Next let n be a number such that $(x_1 + 1 \, , \, y_1 + n)$ is a point on p . Then the slope of p is

$$\frac{(y_1 + n) - y_1}{(x_1 + 1) - x_1} = n \, .$$

Since the slope is m by hypothesis, it follows that $m = n$. Therefore (x_1,y_1) and $(x_1 + 1 \, , \, y_1 + m)$ are two points on ℓ and it follows from Theorem 8-11 that

(2) $p = \{(x,y) : x = x_1 + k \, , \; y = y_1 + km \, , \; k \text{ is real}\}$.

We now consider three possibilities for the slope of a line; it is positive, it is zero, or it is negative. Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be two points of a line. We



suppose that the points are named so that P_2 has the larger x-coordinate. We disregard the possibility $x_2 = x_1$, since this would imply that $\overleftrightarrow{P_1P_2}$ is a vertical line.

__Possibility 1__. The slope is positive. Then $y_2 - y_1$ and $x_2 - x_1$ are both positive or both negative. Since we named the points so that $x_2 > x_1$ it follows that $x_2 - x_1$ and $y_2 - y_1$ are both positive. This means intuitively that, as a particle moves along $\overleftrightarrow{P_1P_2}$ from left to right (from the point with x-coordinate x_1 to the point with x-coordinate x_2), it is going uphill.

__Possibility 2__. The slope is zero. Then $y_2 - y_1 = 0$. This means intuitively that, as a particle moves along the line $\overleftrightarrow{P_1P_2}$ it is moving on "level ground." (The y-coordinates of all the points of the line are the same.)

__Possibility 3__. The slope is negative. Then one of the numbers, $y_2 - y_1$ and $x_2 - x_1$, is positive and the other one is negative. Since we named the points so that $x_2 > x_1$ it follows that $x_2 - x_1$ is positive and $y_2 - y_1$ is negative, that is, $y_2 < y_1$. This means intuitively that, as a particle moves along $\overleftrightarrow{P_1P_2}$ from left to right, it is going downhill.

This section closes with several examples involving the slope idea.

Example 1.

Given $A = (5,-8)$ and $B = (-2,8)$. Find $m_{\overleftrightarrow{AB}}$.

Solution: $m_{\overleftrightarrow{AB}} = \frac{8 - (-8)}{-2 - 5} = -\frac{16}{7}$.

Example 2. A line r passes through $(1,3)$ and has slope 5. Find the point on r whose x-coordinate is -3.

Solution: $r = \{(x,y) : x = 1 + k\ ,\ y = 3 + 5k\ ,\ k \text{ is real}\}$
Set $x = -3$. Then $-3 = 1 + k$, $k = -4$; $y = 3 - 20 = -17$.
Answer: $(-3,-17)$.

Example 3. Find the slope of the line
$\{(x,y) : x = 3 + 4k\ ,\ y = 2\ ,\ k \text{ is real}\}$.

Solution: Set $k = 0$ and 1 to get two points on the line.

$k = 0$: $(x,y) = (3,2)$;
$k = 1$: $(x,y) = (7,2)$.

Then the slope is $\frac{2 - 2}{7 - 3} = 0$.

Alternate solution: By inspection of the parametric equations, $x_2 - x_1 = 4$ and $y_2 - y_1 = 0$. Therefore $m = \frac{0}{4} = 0$.

Problem Set 8-8a

1. Find the slope of the segment joining the points in each of the following pairs.

   (a) $(0,0)$ and $(6,2)$.
   (b) $(0,0)$ and $(6,-2)$.
   (c) $(3,5)$ and $(7,12)$.
   (d) $(0,0)$ and $(-4,-3)$.
   (e) $(-5,7)$ and $(3,-8)$.
   (f) $(\frac{1}{2},3)$ and $(3,\frac{1}{2})$.
   (g) $(-2.8,4)$ and $(4.2,-1)$.
   (h) $(\frac{1}{3},0)$ and $(0,-\frac{1}{4})$.
   (i) $(1000,-500)$ and $(1001,-499)$
   (j) (a,b) and (b,a); $(a \neq b)$.

2. Replace the "?" by a number so that the line through the two points will have the slope given. (Hint: Substitute in the slope formula.)

   (a) (5,2) and (?,6) , $m = 4$.
   (b) (-3,1) and (4,?) , $m = \frac{1}{2}$.
   (c) (6,-3) and (9,?) , $m = -\frac{2}{3}$.
   (d) (?,12) and (5,12) , $m = 0$.

3. Plot the points $A(-1,0)$, $B(6,2)$, $C(4,5)$, $D(-3,3)$. Draw ABCD . Find the slope of each side of ABCD . Which two sides have the same slope?

4. Plot the quadrilateral PQRS with vertices $P(0,4)$, $Q(2,3)$, $R(-1,-2)$, $S(-3,-1)$. Which pairs of sides have the same slope?

5. Without plotting tell whether the slope of the segment joining the points in each of the following pairs has a positive, zero, or negative slope. Then tell how you would interpret the sign of a slope.

   (a) (-3,4) and (2,0) . (d) (3,2) and (5,0) .
   (b) (-3,4) and (2,4) . (e) (5,0) and (3,2) .
   (c) (-3,4) and (2,8) . (f) (-1,4) and (0,10) .

6. Which of the segments joining the points in each of the following pairs is steeper?

   (0,0) and (100,101) or (0,0) and (101,100) ?

7. Find the slope of the line segment joining $(a,\frac{a}{b})$ and $(b,\frac{b}{a})$ if $a \neq b$.

8. Given: $\overleftrightarrow{AB} = \{(x,y) : x = 3 - 2k,\ y = -1 + 3k,\ k \text{ is real}\}$
   What is the slope of $\overleftrightarrow{AB}$?

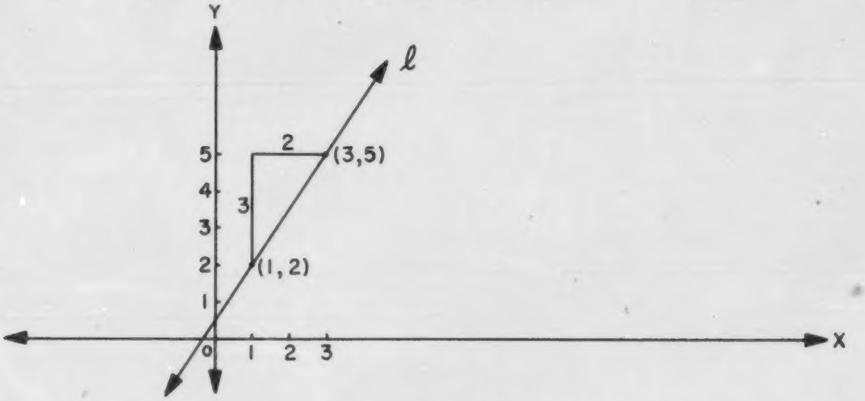
9. Parametric equations of a line are useful in plotting the graph of a line when one point and the slope are given. Consider, for example, the line ℓ through $P(1,2)$ with slope $\frac{3}{2}$.

   $$\ell = \{(x,y) : x = 1 + 2k,\ y = 2 + 3k,\ k \text{ is real}\} .$$

   If $k = 0$, then $(x,y) = (1,2)$.
   If $k = 1$, then $(x,y) = (3,5)$.

If $k = 2$, then $(x,y) = (5,8)$.
If $k = 3$, then $(x,y) = (7,11)$.

Note that as k is assigned values 0, 1, 2, 3, ... , (successive increases of 1), the corresponding x-values are 1, 3, 5, 7, ... (successives increases of $\underline{2}$), and the corresponding y-values are 2, 5, 8, 11, ... , (successive increases of $\underline{3}$). Note that 2 and 3 are the coefficients of k in the parametric equations, and that $\frac{3}{2}$ is the slope. The numerator and denominator of the "slope fraction" tell us how to get from one point to another on the line as suggested in the figure.



Use this method to plot the lines determined in each of the following:

(a) $P_1 = (-3,2)$; slope $= \frac{2}{3}$.

(b) $P_1 = (0,0)$; slope $= \frac{3}{5}$.

(c) $P_1 = (2,-4)$; slope $= -\frac{4}{3}$.

(d) $P_1 = (-1,-3)$; slope $= 2$.

(e) $P_1 = (0,0)$; slope $= \frac{b}{a}$.

10. Plot the graph of lines through the origin having the following slopes:

(a) $\frac{3}{2}$.

(b) $-\frac{3}{5}$.

(c) 4 .

(d) $\frac{r}{2}$; $r < 0$.

11. Write the parametric equations for the lines in Problem 10.

Our next theorem gives us a relation between the concept of parallel lines and the concept of slope.

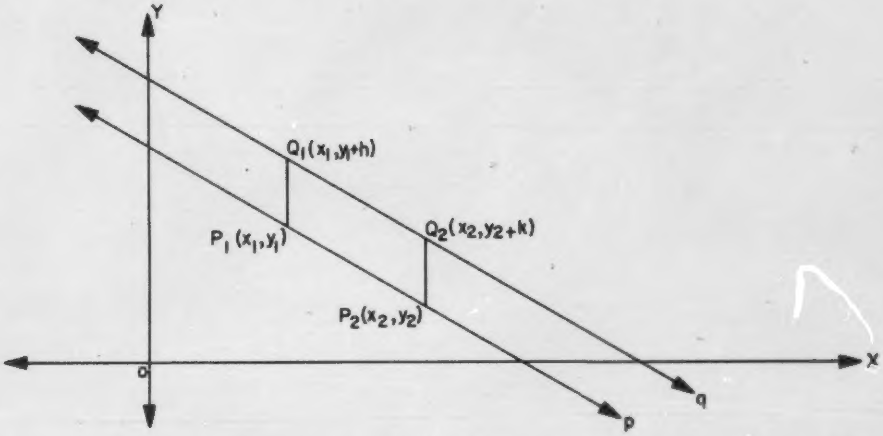
THEOREM 8-15. Two non-vertical lines are parallel if and only if their slopes are equal.

Proof: Let two distinct non-vertical lines p and q be given. We have two things to prove:

(1) If $p \parallel q$, then their slopes are equal.

(2) If the slopes of p and q are equal, then $p \parallel q$.

(1) Suppose $p \parallel q$.



Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be two points in p . Let vertical lines through P_1 and P_2 intersect q in $Q_1(x_1,y_1+h)$ and $Q_2(x_2,y_2+k)$, respectively. Then $P_1Q_1Q_2P_2$ is a parallelogram. Therefore $P_1Q_1 = P_2Q_2$. Since $P_1Q_1 = |h|$, $P_2Q_2 = |k|$, and since h and k are both positive or both negative, it follows that $h = k$.

But the slope of p is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

and the slope of q is

$$\frac{(y_2 + k) - (y_1 + h)}{x_2 - x_1} = \frac{y_2 + h - y_1 - h}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1} .$$

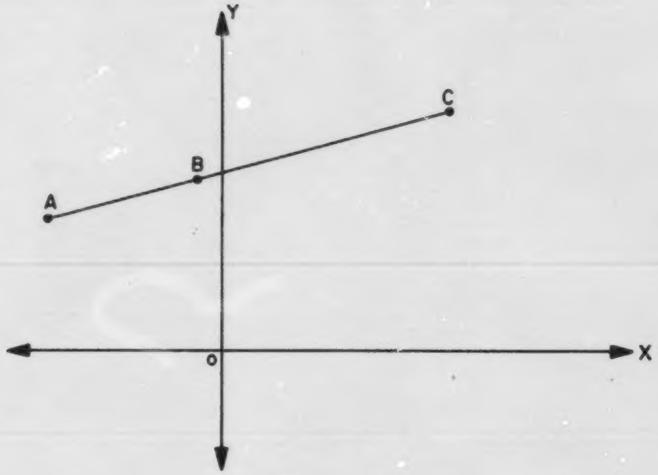
Therefore the slopes are equal.

(2) Next, suppose there is a number m which is the slope of both p and q. We want to prove that the lines are parallel. We do this by showing that if they have one point in common, then they are the same line. Suppose then that they have a point, say $R(x_1,y_1)$, in common. Since p is not vertical, it contains a point $P(x_2,y_2)$ such that $x_2 \neq x_1$. Since q is not vertical, it intersects the line $\{(x,y) : x = x_2\}$; that is, q contains a point $Q(x_3,y_3)$ such that $x_3 = x_2$. Since the slopes of $\overline{PR}$ and $\overline{QR}$ are the same,

$$\frac{y_2 - y_1}{x_2 - x_1} = m = \frac{y_3 - y_1}{x_3 - x_1} .$$

Since $x_2 = x_3$, the denominators $x_2 - x_1$ and $x_3 - x_1$, are the same. Hence $y_2 - y_1 = y_3 - y_1$, or $y_2 = y_3$. This means that $Q = P$. In other words, if p and q intersect, then p and q are the same line and therefore parallel. If p and q do not intersect then they are parallel by definition. This completes the proof that if the slopes of p and q are equal, then p and q are parallel.

A natural question to ask is the following one. If the slopes of two segments are equal, and have a point in common, are they collinear? This suggests the test for collinearity stated in the next corollary.



Corollary 8-15. Three points A, B, C are collinear if and only if $m_{\overline{AB}} = m_{\overline{BC}}$, or they lie on a vertical line.

This "if and only if" statement is a short statement combining the two statements:

(1) If $m_{\overline{AB}} = m_{\overline{BC}}$, then A, B, C are collinear.

(2) If A, B, C are collinear, and do not lie on a vertical line, then $m_{\overline{AB}} = m_{\overline{BC}}$.

Proof: Let A, B, C be three points and let $m_1 = m_{\overline{AB}}$, $m_2 = m_{\overline{BC}}$. Then

$$\overleftrightarrow{BA} = \{(x,y) : x = x_B + k \ , \ y = y_B + km_1 \ , \ k \text{ is real}\} .$$

$$\overleftrightarrow{BC} = \{(x,y) : x = x_B + k \ , \ y = y_B + km_2 \ , \ k \text{ is real}\} .$$

If $m_1 = m_2$, then $\overleftrightarrow{BA} = \overleftrightarrow{BC}$ and A, B, C are collinear. If A, B, C are collinear, then it follows directly from Theorem 8-15 that $m_1 = m_2$.

Problem Set 8-8b

1. Show that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and that $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ if
   (a) $A = (-3,-2)$, $B = (5,1)$, $C = (6,6)$, $D = (-2,3)$.
   (b) $A = (5,-3)$, $B = (15,-2)$, $C = (26,-2)$, $D = (16,-3)$.
   (c) $A = (-3,0)$, $B = (1,5)$, $C = (10,2)$, $D = (6,-3)$.

2. Show that $\overline{AB}$ is not parallel to $\overline{CD}$ if $A = (6,2)$, $B = (-1,4)$, $C = (-1,2)$, $D = (8,0)$.

3. (a) Is the point $B(4,13)$ on the line joining $A(1,1)$ to $C(5,17)$?
   (b) Is the point $(2,-1)$ collinear with $(-5,4)$ and $(6,-8)$?
   (c) Given: $A = (101,102)$, $B = (5,6)$ and $C = (-95,-94)$. Determine whether $\overleftrightarrow{AB} = \overleftrightarrow{BC}$.
   (d) Given: $A = (101,102)$, $B = (5,6)$, $C = (202,203)$ and $D = (203,204)$. Are $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ parallel? Are they equal?

4. (a) Given: $A = (3,8)$ and the slope of line p containing A is $\frac{2}{3}$. Find the coordinates of three more points on p .
   (b) Given $B = (-1,0)$ and the slope of line q containing B is $-\frac{3}{4}$. Find the coordinates of three more points on q .

5. (a) Write a pair of parametric equations of the line containing $(3,4)$ whose slope is $\frac{2}{3}$.
   (b) Write a pair of parametric equations of the line containing $(-1,3)$ whose slope is -1 . (Hint: $-1 = \frac{-1}{1}$.)

6. Given: $\overleftrightarrow{AB} = \{(x,y) : x = 3 - 2k , y = -1 + 3k , k \text{ is real}\}$ What is the slope of $\overleftrightarrow{AB}$? Express $\overleftrightarrow{CD}$ in parametric equations if $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ contains $(0,0)$.

7. Given $a = \{(x,y) : x = 1 + 2k , y = 2 - k , k \text{ is real}\}$,
   $b = \{(x,y) : x = 3 + 2h , y = -1 - h , h \text{ is real}\}$,
   show that $a \parallel b$. As part of your proof, show that $a \neq b$.

8. Given $p = \{(x,y) : x = 1 + 2k,\ y = 3 + 4k,\ k \text{ is real}\}$,
$q = \{(x,y) : x = 1 - 4h,\ y = 3 - 8h,\ h \text{ is real}\}$.
Show that $p = q$.

9. Given $m = \{(x,y) : x = 1 + 2k,\ y = 2 + 3k,\ k \text{ is real}\}$,
$n = \{(x,y) : x = 1 - 2h,\ y = 2 + 3h,\ h \text{ is real}\}$.

   (a) Show that m and n intersect in one point.
   (b) Find the coordinates of that point.

10. Four points taken in pairs determine six segments. Which pairs of distinct segments determined by the following four points are parallel? $A(3,6)$, $B(5,9)$, $C(8,2)$, $D(6,-1)$.

11. Show by considering slopes that a parallelogram is formed by drawing segments joining in order $A(-1,5)$, $B(5,1)$, $C(6,-2)$ and $D(0,2)$.

12. Show that if one of two parallel lines is vertical, then the other is also.

13. Given $A(-2,-4)$, $B(4,2)$, $C(6,0)$. Let D be the midpoint of $\overline{AB}$ and E the midpoint of $\overline{BC}$. Show that $\overline{DE}$ is parallel to $\overline{AC}$.

14. It is asserted that both of the quadrilaterals whose vertices are given below are parallelograms. Without plotting the points determine whether or not this is true.

   (a) $A(-5,-2)$, $B(-4,2)$, $C(4,6)$, $D(3,1)$.
   (b) $P(-2,-2)$, $Q(4,2)$, $R(9,1)$, $S(3,-3)$.

15. Show that the line through $(3n,0)$ and $(0,n)$ is parallel to the line through $(6n,0)$ and $(0,2n)$. Assume $n \neq 0$.

16. $P = (a,1)$, $Q = (3,2)$, $R = (b,1)$, $S = (4,2)$. Prove that $\overleftrightarrow{PQ} \neq \overleftrightarrow{RS}$ and that $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$ if and only if $a = b - 1$.

8-9. Other Equations for Lines.

In the preceding sections we have used parametric equations to express the coordinates of the points of a line. In this section we find another expression which "represents" a line. We illustrate with a particular line.

Consider the line $\overleftrightarrow{AB}$ where $A = (2,5)$ and $B = (4,8)$. Then $P(x,y)$ is collinear with A and B if and only if $P = A$ or $m_{\overline{AP}} = m_{\overline{AB}}$. Since $m_{\overline{AP}} = \frac{y-5}{x-2}$ and $m_{\overline{AB}} = \frac{8-5}{4-2} = \frac{3}{2}$, $P(x,y)$ is collinear with A and B if and only if $\frac{y-5}{x-2} = \frac{3}{2}$, or $(x,y) = (2,5)$. If $\frac{y-5}{x-2} = \frac{3}{2}$, then $\frac{x-2}{2} = \frac{y-5}{3}$. If $(x,y) = (2,5)$ then $x - 2 = 0$, $y - 5 = 0$, and $\frac{x-2}{2} = \frac{y-5}{3}$. Conversely if $\frac{x-2}{2} = \frac{y-5}{3}$ then $\frac{y-5}{x-2} = \frac{3}{2}$ or $(x,y) = (2,5)$. It follows that $P(x,y)$ is collinear with A and B if and only if $\frac{x-2}{2} = \frac{y-5}{3}$. Therefore

$$\overleftrightarrow{AB} = \left\{(x,y) : \frac{x-2}{2} = \frac{y-5}{3}\right\}.$$

If we think of A as (x_A,y_A) and B as (x_B,y_B), the expression appears as

$$\overleftrightarrow{AB} = \left\{(x,y) : \frac{x-x_A}{x_B-x_A} = \frac{y-y_A}{y_B-y_A}\right\}.$$

This suggests the following theorem.

THEOREM 8-16. If $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ and if $\overleftrightarrow{PQ}$ is an oblique line, then

$$\overleftrightarrow{PQ} = \left\{(x,y) : \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}\right\}.$$

Proof: The point $R(x,y)$ is collinear with P and Q if and only if $R = P$ or $m_{\overleftrightarrow{PR}} = m_{\overleftrightarrow{PQ}}$, that is, if and only if $(x,y) = (x_1,y_1)$ or

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \ ;$$

that is, if and only if

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \ .$$

Corollary 8-16-1. If $\overleftrightarrow{PQ}$ is the line of Theorem 8-16, then

$$\overleftrightarrow{PQ} = \{(x,y) : y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\} \ .$$

Proof: To prove this we show that the equation of the theorem, $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$, is equivalent to the equation of the corollary,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \ .$$

To get the second form from the first, multiply both sides of the first by $y_2 - y_1$; to get the first from the second divide both sides of the second by $y_2 - y_1$.

Corollary 8-16-2. If p is the line which passes through $P(x_1,y_1)$ with slope m , then

$$p = \{(x,y) : y - y_1 = m(x - x_1)\}.$$

Proof: If $m \neq 0$, this follows immediately from Corollary 8-14-1 since p also passes through $(x_2,y_2) = (x_1 + 1,\ y_1 + m)$ and then

$$m = \frac{y_2 - y_1}{x_2 - x_1}\ .$$

If $m = 0$, the equation reduces to $y = y_1$.

DEFINITION. The equation $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1}$ is called the *two-point form* for the equation of an oblique line $\overleftrightarrow{P_1P_2}$, where $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$.

DEFINITION. The equation $y - y_1 = m(x - x_1)$ is called the *point slope form* for the equation of a non-vertical line with slope m and passing through (x_1,y_1).

Example 1. If $C = (7,-3)$ and $D = (4,-5)$, then

$$\overleftrightarrow{CD} = \left\{(x,y) : \frac{x-7}{4-7} = \frac{y+3}{-5+3}\right\} = \left\{(x,y) : \frac{x-7}{-3} = \frac{y+3}{-2}\right\}$$

$$= \left\{(x,y) : \frac{x-7}{3} = \frac{y+3}{2}\right\}\ .$$

Example 2. Write an equation for the line through $(-5,-2)$ with slope 4. Answer: $y + 2 = 4(x + 5)$.

Example 3. If $A = (2,1)$, $B = (3,4)$, $C = (5,-2)$, $D = (0,-1)$, find the point of intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Solution:

$\overleftrightarrow{AB} = \{(x,y) : \frac{x - 2}{3 - 2} = \frac{y - 1}{4 - 1}\} = \{(x,y) : 3x - y = 5\}$.

$\overleftrightarrow{CD} = \{(x,y) : \frac{x - 5}{0 - 5} = \frac{y + 2}{-1 + 2}\} = \{(x,y) : x + 5y = -5\}$.

Also $m_{\overleftrightarrow{AB}} = \frac{4 - 1}{3 - 2} = 3$, $m_{\overleftrightarrow{CD}} = \frac{-1 + 2}{0 - 5} = -\frac{1}{5}$. Since these slopes are unequal the lines intersect in some point (x_1,y_1) . Then $3x_1 - y_1 = 5$, $x_1 + 5y_1 = -5$. Multiplying both sides of the first equation by 5 and adding to the sides of the second equation we get $16x_1 = 20$. Then $x_1 = \frac{5}{4}$, $y_1 = -\frac{5}{4}$. Therefore $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in the point $(\frac{5}{4} , -\frac{5}{4})$.

Problem Set 8-9

1. Write an equation in two-point form for the line determined by the given pair of points.

   (a) (1,4) and (4,3) .
   (b) (0,5) and (-3,0) .
   (c) (0,-5) and (3,0) .
   (d) (-3,2) and (5,-4) .
   (e) (0,0) and (7,-8) .
   (f) (-1,1) and (1,-1) .

2. Write an equation in point-slope form for the line which contains the given point and has the given slope.

   (a) (0,0) , $\frac{1}{2}$.
   (b) (-3,5) , $\frac{2}{3}$.
   (c) (-2,7) , $-\frac{3}{4}$.
   (d) (-3,-2) , 2 .
   (e) (-3,2) , -1 .
   (f) (0,-5) , 3 .

3. Write an equation in point-slope form of the line that contains the given point (5,8) and is parallel to the line found in Problem 2(c).

4. In triangle ABC , A = (0,0) , B = (1,6) , and C = (5,2) .

   (a) Write an equation for $\overleftrightarrow{AB}$.
   (b) Write an equation for $\overleftrightarrow{AC}$.
   (c) Write an equation for the line that contains the median from A .

(d) Write an equation for the line that contains the midpoints of $\overline{AB}$ and $\overline{AC}$.

(e) Write an equation for the line $\overleftrightarrow{BC}$.

(f) If $D = (7,7)$, find the coordinates of the point of intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$.

5. Below are equations of lines. Which of these lines contains $(2,3)$?

(a) $2y = 3x$.

(b) $3y = 2x$.

(c) $y - 3 = 2(x - 2)$.

(d) $y = 2x - 1$.

(e) $\frac{x}{2} + \frac{y}{3} = 1$.

(f) $\frac{y - 2}{x - 3} = \frac{3}{2}$.

6. Write an equation of the line that contains $(-2,4)$ and whose slope is the given number.

(a) 2 .

(b) 1 .

(c) 0 .

(d) -1 .

(e) $\frac{1}{2}$.

(f) $-\frac{3}{2}$.

7. Given below is a set of four lines. State which pairs of lines are parallel.

$p = \{(x,y) : x - 2y = 8\}$, $q = \{(x,y) : 2x + y = 1\}$,
$r = \{(x,y) : 4x + 2y = 3\}$, $s = \{(x,y) : 2x - 4y = 11\}$

8. For each pair p and q determine whether $p \parallel q$, p and q intersect in one point, or $p = q$.

(a) $p = \{(x,y) : x - 2y = 8\}$ and $q = \{(x,y) : 2x + y = 1\}$

(b) $p = \{(x,y) : x - 2y = 8\}$ and $q = \{(x,y) : 2x - 4y = 16\}$

(c) $p = \{(x,y) : x - 2y = 8\}$ and $q = \{(x,y) : 2x - 4y = 10\}$

9. Given two non-zero numbers a and b , show that $\frac{x}{a} + \frac{y}{b} = 1$ is an equation of the line that contains $(a,0)$ and $(0,b)$. This form of a linear equation is called the intercept form.

10. Given two numbers m and b , show that $y = mx + b$ is an equation of the line whose slope is m and which intercepts the y-axis at a point whose y-coordinate is b . This form of a linear equation is called the slope-intercept form.

8-10. Perpendicular Lines.

We have seen that two non-vertical lines are parallel if and only if their slopes are equal. In this section we develop a condition in terms of slopes for the perpendicularity of two lines. If one of two lines is vertical, then a necessary and sufficient condition that the lines be perpendicular is that the other one be horizontal. The following theorem is a statement about the perpendicularity of two non-vertical lines in terms of their slopes.

THEOREM 8-17. Two non-vertical lines are perpendicular if and only if the product of their slopes is -1 .

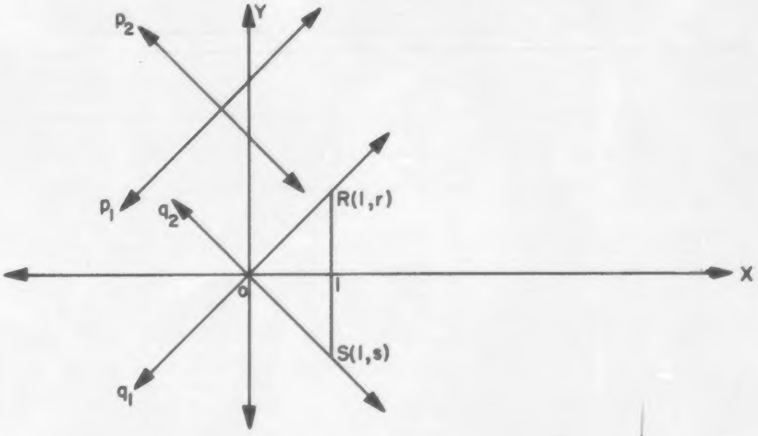
Proof: Let the given lines be denoted by p_1 and p_2 and let their slopes be m_1 and m_2 , respectively. We have two statements to prove.

(1) If $p_1 \perp p_2$, then $m_1 m_2 = -1$.

(2) If $m_1 m_2 = -1$, then $p_1 \perp p_2$.

We prove both statements together as follows.

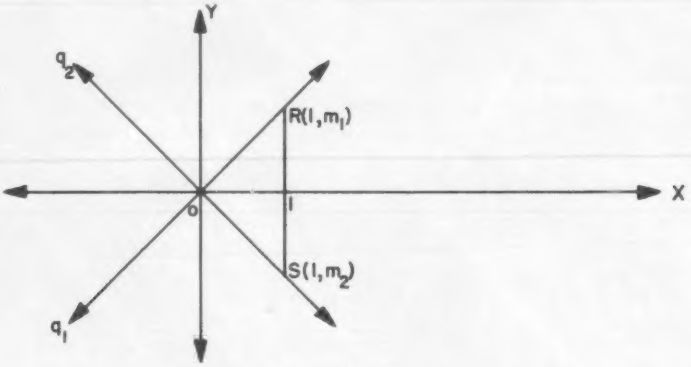
Let q_1 be the line containing $(0,0)$ which is parallel to p_1 . Let q_2 be the line containing $(0,0)$ which is parallel to p_2 . The slope of q_1 is m_1 and the slope of q_2 is m_2 . Let q_1 and q_2 intersect the vertical line $\{(x,y) : x = 1\}$ in $R(1,r)$ and $S(1,s)$ respectively.



Then $m_1 = m_{\overline{OR}} = \frac{r - 0}{1 - 0} = r$,

$m_2 = m_{\overline{OS}} = \frac{s - 0}{1 - 0} = s$.

Therefore $R = (1,m_1)$ and $S = (1,m_2)$.



Then $p_1 \perp p_2$ if and only if $q_1 \perp q_2$, and $q_1 \perp q_2$ if and only if $\overline{OR} \perp \overline{OS}$.

From the Pythagorean Theorem and its converse it then follows that

$$p_1 \perp p_2 \text{ if and only if } (OR)^2 + (OS)^2 = (RS)^2 .$$

Using the distance formula we get

$$(OR)^2 = 1 + m_1^{\,2} , \ (OS)^2 = 1 + m_2^{\,2} , \ (RS)^2 = (m_1 - m_2)^2 .$$

Then $p_1 \perp p_2$ if and only if $1 + m_1^{\,2} + 1 + m_2^{\,2} = (m_1 - m_2)^2$,

if and only if $2 + m_1^{\,2} + m_2^{\,2} = m_1^{\,2} - 2m_1m_2 + m_2^{\,2}$,

if and only if $2 = -2m_1m_2$,

if and only if $m_1m_2 = -1$,

which completes the proof.

Example 1. Given $A = (2,5)$, $B(-1,7)$, $C = (4,1)$, $D = (8,7)$, show that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

Solution: $m_{\overleftrightarrow{AB}} = \frac{7-5}{-1-2} = -\frac{2}{3}$

$$m_{\overleftrightarrow{CD}} = \frac{7-1}{8-4} = \frac{6}{4} = \frac{3}{2} .$$

Since $-\frac{2}{3} \cdot \frac{3}{2} = -1$ it follows that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

Example 2. Given $P = (4,-15)$, $Q = (-17,3)$, $R = (0,5)$, determine whether or not $\overline{PQ}$ is perpendicular to $\overline{QR}$.

Solution: $m_{\overleftrightarrow{PQ}} = \frac{3+15}{-17-4} = \frac{18}{-21} = -\frac{6}{7}$, $m_{\overleftrightarrow{QR}} = \frac{3-5}{-17} = \frac{2}{17}$,

$m_{\overleftrightarrow{PQ}} \cdot m_{\overleftrightarrow{QR}} = \frac{-12}{119} \neq -1$. Therefore $\overleftrightarrow{PQ}$ is not perpendicular to $\overleftrightarrow{QR}$.

Example 3. If $A = (0,0)$, $B = (4,3)$, $C = (8,9)$, $D = (-5,11)$, prove that the diagonals of quadrilateral ABCD are perpendicular.

Solution: Since $m_{\overline{AC}} = \frac{9}{8}$, $m_{\overline{BD}} = \frac{11-3}{-5-4} = -\frac{8}{9}$, and since $\frac{9}{8} \cdot (\frac{-8}{9}) = -1$, it follows that $\overline{AC} \perp \overline{BD}$.

Example 4. Given $A = (5,-7)$, $B = (0,0)$, $C = (7,5)$, determine whether or not triangle ABC is a right triangle.

Solution: Since $m_{\overline{AB}} = -\frac{7}{5}$, $m_{\overline{AC}} = 6$, $m_{\overline{BC}} = \frac{5}{7}$, it follows that $\overline{AB} \perp \overline{BC}$ and hence ABC is a right triangle with right angle at B .

Example 5. Given $A = (-1,3)$, and $B = (5,-1)$, find an equation in point-slope form for the perpendicular bisector of $\overline{AB}$ in the xy-plane.

Solution: Since the slope of $\overleftrightarrow{AB} = \frac{-1-3}{5+1} = -\frac{2}{3}$, it follows that the slope of the perpendicular bisector is $\frac{3}{2}$. The midpoint of $\overline{AB}$ is $\left(\frac{-1+5}{2}, \frac{3-1}{2}\right) = (2,1)$. Then the equation of the perpendicular bisector is $y - 1 = \frac{3}{2}(x - 2)$, or $3x - 2y = 4$.

Alternate solution: The midpoint of $\overline{AB}$ is $(2,1)$ and the slope of the perpendicular bisector of $\overline{AB}$ is $\frac{3}{2}$ as in the above solution. Then parametric equations for the perpendicular bisector are

$$x = 2 + 2k, \quad y = 1 + 3k.$$

Then $3x = 6 + 6k$, $2y = 2 + 6k$; $3x - 6 = 2y - 2$; $3x - 2y = 4$. It follows that $3x - 2y = 4$ is an equation for the perpendicular bisector of $\overline{AB}$ in the xy-plane.

Problem Set 8-10

1. Lines p, q, r, and s have slopes $\frac{2}{3}$, -4, $-1\frac{1}{2}$, and $\frac{1}{4}$, respectively. Which pairs of lines are perpendicular?
2. The vertices of a triangle are $A(16,0)$, $B(9,2)$, and $C(0,0)$.
   - (a) What is the slope of $\overline{AB}$?
   - (b) What is the slope of a line that is perpendicular to $\overline{AB}$?
   - (c) What is the slope of $\overline{BC}$?
   - (d) What is the slope of a line that is perpendicular to $\overline{BC}$?
3. Show that the line containing $(0,0)$ and $(3,2)$ is perpendicular to the line containing $(0,0)$ and $(-2,3)$.

4. Show that $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$ if $A = (a,b)$, $B = (0,0)$, and $C = (-b,a)$ where $a \neq 0$ and $b \neq 0$.

5. Given the points $P(1,2)$, $Q(5,-6)$, and $R(b,b)$, determine the value of b so that $\angle PQR$ is a right angle.

6. Given $\overleftrightarrow{AB} = \{(x,y) : x = 1 + 2k \text{ and } y = 2 + 3k, k \text{ is real}\}$, and $\overleftrightarrow{CD} = \{(x,y) : x = 1 - 3k \text{ and } y = 2 + 2k, k \text{ is real}\}$.

   Prove: (1) $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in $(1,2)$.
   (2) $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

7. Given $\overleftrightarrow{AB} = \{(x,y) : x = -1 + 4k, y = 2 - 3k, k \text{ is real}\}$ If $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ contains $(-2,2)$, express $\overleftrightarrow{CD}$ with set notation symbols and parametric equations.

8. The vertices of triangle ABC are $A(0,0)$, $B(3,2)$, and $C(4,-1)$. Using parametric equations express:
   - (a) The line through B that is parallel to $\overleftrightarrow{AC}$.
   - (b) The line through B that is perpendicular to $\overleftrightarrow{AC}$.
   - (c) The line through A that is parallel to $\overleftrightarrow{BC}$.
   - (d) The line through A that is perpendicular to $\overleftrightarrow{BC}$.
   - (e) The coordinates of D if D is on $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$.

9. Using slopes, show that the quadrilateral $A(8,0)$, $B(6,4)$, $C(-2,0)$, $D(0,-4)$ has four right angles.

10. Express in set notation the perpendicular bisector of the segment that joins the following pairs of points.
    - (a) $(3,3)$ and $(1,1)$.
    - (b) $(-3,2)$ and $(3,-1)$.
    - (c) $(-3,-1)$ and $(3,5)$.
    - (d) $(0,0)$ and (a,b).

11. Show that if (a,b) and (c,d) are distinct points, the line p containing them is perpendicular to the line q joining (b,c) to (d,a).

12. Given $A = (0,0)$, $B = (4,2)$, $C = (3,-3)$, $D = (x,-2)$.
    - (a) Find x so that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.
    - (b) Find x so that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

13. Given: $A(3,5)$ and $B(-1,-2)$. Calculate the slope of the line perpendicular to $\overline{AB}$ through A and draw the line.

14. Given $P(-3,1)$, $Q(0,-5)$, and $R(5,0)$. Calculate $m_{\overline{PQ}}$.
    (a) Through R draw a line parallel to $\overleftrightarrow{PQ}$.
    (b) Through R draw a line perpendicular to $\overleftrightarrow{PQ}$.

15. The slope of a line p through $(2,3)$ is $\frac{2}{3}$.
    (a) Give the coordinates of two other points on p.
    (b) Give the coordinates of two other points which are contained in a line through $(2,3)$ perpendicular to p.

16. Given a quadrilateral $A(a,b)$, $B(a+c,b)$, $C(a+c,b+c)$, $D(a,b+c)$.
    (a) Prove that $\overline{AC} \cong \overline{BD}$.
    (b) Prove that $\overline{AC} \perp \overline{BD}$.
    (c) Prove that $\overline{AC}$ and $\overline{BD}$ have the same midpoint.

8-11. Parallelograms.

This section contains several definitions and theorems relating to parallelograms. In Chapter 6 we defined a parallelogram as a quadrilateral each of whose sides is parallel to the side opposite it and proved two theorems. They are

1. In any parallelogram each side is congruent to the side opposite it. (Theorem 6-6)
2. If two sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram. (Theorem 6-7)

In Problems 2 and 5 of Problem Set 6-7 and Problem 5 of Problem Set 6-8b, we proved statements which we now introduce formally as theorems.

THEOREM 8-18. A quadrilateral is a parallelogram if each of its sides is congruent to the side opposite it.

THEOREM 8-19. A quadrilateral is a parallelogram if and only if each angle is congruent to the angle opposite it.

We now consider cases of special parallelograms which have properties not common to all parallelograms.

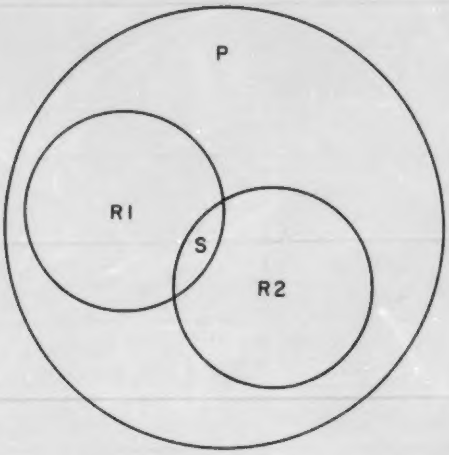
DEFINITION. A parallelogram is a rectangle if and only if it has a right angle.

Perhaps you think of a rectangle as a quadrilateral having four right angles. It is possible to start with this as a definition or the one given above. In either case the other statement becomes a theorem.

DEFINITION. A parallelogram is a rhombus if and only if two consecutive sides are congruent.

DEFINITION. A parallelogram is a square if and only if it has a right angle and two adjacent sides that are congruent.

You should notice that every square is a rectangle and also a rhombus. We might say that the set of squares is the intersection of the set of rectangles and rhombuses. We can picture roughly the set relations as follows:



In this diagram the region marked P represents the set of parallelograms; the region marked R_1, the set of rectangles; the region R_2, the set of rhombuses; the region marked S, the set of squares.

The following theorem is a direct consequence of our definition of a rectangle and Theorem 8-19.

THEOREM 8-20. A quadrilateral is a rectangle if and only if it is equiangular.

The proof is left as a problem.

As a direct consequence of the definition of a rhombus and Theorem 8-18, we also prove:

THEOREM 8-21. A quadrilateral is a rhombus if and only if it is equilateral.

Problem Set 8-11

1. Does a rhombus have all the properties of a parallelogram? Does a parallelogram have all the properties of a rhombus? Explain.
2. Define a square in terms of: (a) a rhombus,
   (b) a rectangle.

3. (a) Write the two parts of Theorem 8-20.
   (b) Prove both parts of the theorem.

4. (a) Write the two parts of Theorem 8-21.
   (b) Prove both parts of the theorem.

5. Identify the following statements as true or false. Be able to justify your answer for each statement.

   (a) If a quadrilateral is a rectangle, it is equiangular.
   (b) If a quadrilateral is equiangular, it is a rectangle.
   (c) If a quadrilateral is a rhombus, it is equilateral.
   (d) If a quadrilateral is equilateral, it is a rhombus.
   (e) If a quadrilateral is regular, it is a square.
   (f) The two triangles determined by a diagonal of a parallelogram are congruent.
   (g) If the triangles determined by one diagonal of a quadrilateral are congruent, the quadrilateral is a parallelogram.

8-12. Using Coordinates in Proofs.

We have seen that the xy-coordinate system is a useful tool in solving problems in geometry. As we pointed out in the beginning of this chapter, there are many coordinate systems in a plane. It is natural to expect that a coordinate system selected to "fit a problem" might be a better tool than one set up without reference to the problem. And this is indeed the case, as we now illustrate.

Example. Prove that if a line segment joins the midpoints of two sides of a triangle, its length is half the length of the third side.

Proof I:

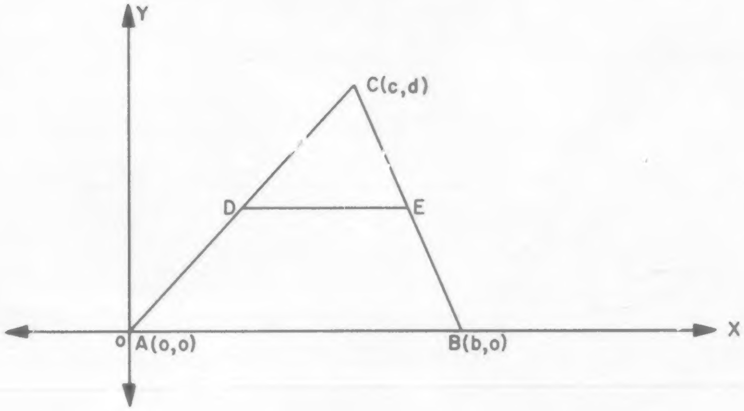
Suppose a triangle and a line segment joining the midpoints of two of its sides are given. Label the given triangle ABC so that the given segment joins the midpoints of sides $\overline{AC}$ and $\overline{BC}$. Call these midpoints D and E respectively.

We now set up an xy-coordinate system in the plane of triangle ABC which seems to fit the problem. We choose line $\overleftrightarrow{OX}$ as the line $\overleftrightarrow{AB}$. We choose point A as the origin. The line $\overleftrightarrow{OY}$ is taken as the line in the plane ABC which is perpendicular to $\overleftrightarrow{OX}$ at A. Then $A = (0,0)$, $B = (b,0)$, $C = (c,d)$, for some real numbers b, c, d. (We know that $b \neq 0$ since A and B are different points. We know that $d \neq 0$, since A, B, C are noncollinear points.)

Then we use the midpoint formula to get

$$D = \left(\frac{0 + c}{2}, \frac{0 + d}{2}\right) = \left(\frac{c}{2}, \frac{d}{2}\right)$$

$$E = \left(\frac{b + c}{2}, \frac{0 + d}{2}\right) = \left(\frac{b + c}{2}, \frac{d}{2}\right).$$



Then $\overleftrightarrow{DE}$ and $\overleftrightarrow{AB}$ are horizontal lines and

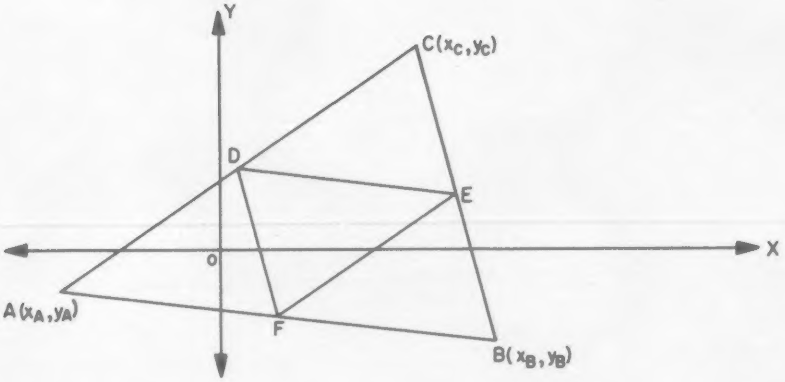
$$DE = \left|\frac{b + c}{2} - \frac{c}{2}\right| = \frac{|b|}{2},$$

$$AB = |b - 0| = |b|.$$

It follows that $DE = \frac{1}{2} AB$, and this completes the proof.

Proof II:

In the above proof we labeled our figure and set up a coordinate system to fit the problem. We now give a proof using a coordinate system which is not set up to fit the problem. [illegible] this proof to see how it compares in difficulty with the above proof.



Suppose a triangle ABC is given and that D, E, F are the midpoints of $\overline{AC}$, $\overline{BC}$, $\overline{AB}$, respectively. Then using the midpoint formula we find that

$$D = \left(\frac{x_A + x_C}{2}, \frac{y_A + y_C}{2}\right), \quad E = \left(\frac{x_B + x_C}{2}, \frac{y_B + y_C}{2}\right).$$

Using the distance formula we get

$$DE = \sqrt{\left(\frac{x_B + x_C}{2} - \frac{x_A + x_C}{2}\right)^2 + \left(\frac{y_B + y_C}{2} - \frac{y_A + y_C}{2}\right)^2}$$

$$= \sqrt{\left(\frac{x_B + x_C - x_A - x_C}{2}\right)^2 + \left(\frac{y_B + y_C - y_A - y_C}{2}\right)^2}$$

$$= \sqrt{\left(\frac{x_B - x_A}{2}\right)^2 + \left(\frac{y_B - y_A}{2}\right)^2}$$

$$= \sqrt{\frac{(x_B - x_A)^2 + (y_B - y_A)^2}{4}}$$

$$= \frac{1}{2}\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}.$$

But

$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}.$$

Therefore $DE = \frac{1}{2} AB$. Similarly $EF = \frac{1}{2} AC$ and $DF = \frac{1}{2} BC$.

As you can see from these two proofs an xy-coordinate system which is set up to fit the problem simplifies the expressions involving coordinates which are used in the proof. In Proof I we found that $y_D = \frac{d}{2} = y_E$. This proves that DE is horizontal and hence that $\overline{DE} \parallel \overline{AB}$.

It might appear in our first proof that we are proving only a special case. Actually the proof applies to all cases. The x-axis and the y-axis can be chosen anywhere in the plane so long as they are perpendicular to each other. For convenience we chose $\overleftrightarrow{AB}$ as the x-axis. We cannot then choose $\overleftrightarrow{AC}$ as the y-axis. For then $\overline{AC} \perp \overline{AB}$ and this would mean that the proof is for the special case of a right triangle. After selecting a line for the x-axis we may select any point in it as the origin. We chose A as the origin. Then $\overleftrightarrow{OY}$ is taken as the unique line in the plane of triangle ABC which is perpendicular to X at A . The proof is general since we can set up such a coordinate system starting with any triangle and the segment joining the midpoints of two of its sides.

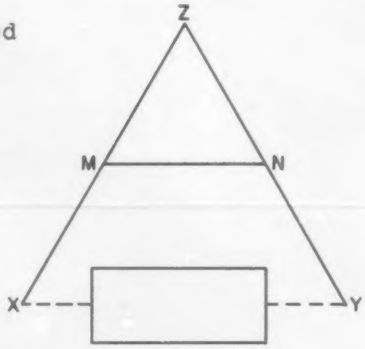
We state as a theorem what we have proved.

THEOREM 8-22. A line segment which joins the midpoints of two sides of a triangle is parallel to the third side and its length is half the length of the third side.

Problem Set 8-12

1. Prove Theorem 8-17 if the coordinates of the vertices of $\triangle ABC$ are: $A = (0,0)$, $B = (2b,0)$, and $C = (2c,2d)$. Is there any advantage in choosing these coordinates rather than the coordinates in Proof I of the example? If there is an advantage, explain.
2. Given $\triangle ABC$ with $AB = 6$, $BC = 8$, and $AC = 10$. Find the perimeter of $\triangle DEF$, if D, E, and F are midpoints of the sides of the triangle.

3. It is desired to measure the distance between two trees on opposite sides of a building. If the two trees are represented by points X and Y , then locate a third point Z from which both X and Y may be seen. Place stakes at M and N , the midpoints of $\overline{XZ}$ and $\overline{YZ}$. How can you find the distance between X and Y after measuring $\overline{MN}$? Explain.



4. In Problem 1, if c = 0 , then $\triangle ABC$ is a __________ triangle. Explain.

*5. Prove that the midpoint of the hypotenuse of a right triangle is equally distant from the vertices of the triangle.

6. Given isosceles triangle ABC . Set up a coordinate system with the vertex of the triangle on the y-axis and the corresponding base of the triangle on the x-axis.

7. Prove the statement: If a triangle is isosceles, the medians to the two congruent sides of the triangle are congruent. [Hint: Let vertices A and B be contained in the x-axis and vertex C be contained in the y-axis.]

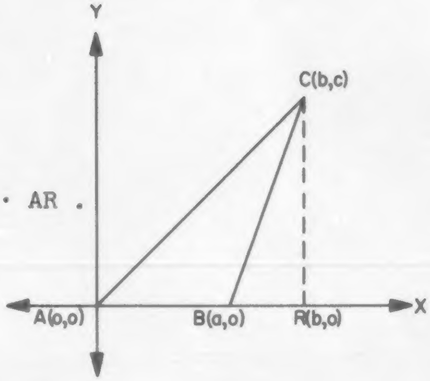
8. Prove the statement: If the medians to two sides of a triangle are congruent, the triangle is isosceles.

9. Write a single statement which combines the statements in Problem 7 and Problem 8.

10. Given: In triangle ABC , $\angle A$ is acute and $\overline{CR}$ is an altitude.

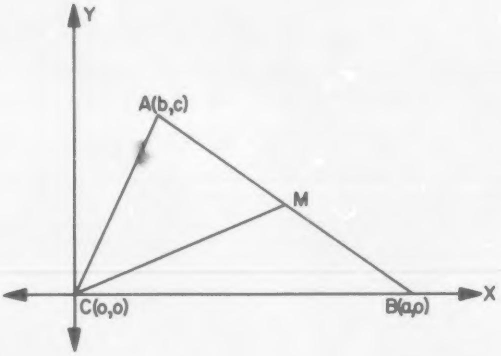
Prove:

$(CB)^2 = (AC)^2 + (AB)^2 - 2AB \cdot AR$.



11. In $\triangle ABC$, $\overline{CM}$ is a median to side $\overline{AB}$. Using coordinates verify:

$(AC)^2 + (BC)^2 = \frac{(AB)^2}{2} + 2(MC)^2$.



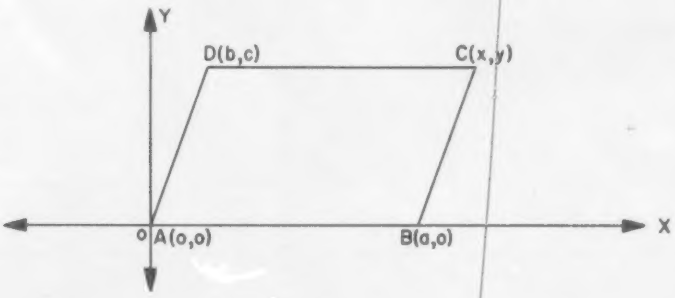
8-13. Parallelograms Continued.

Many theorems relating to parallelograms can be proved with or without coordinates. The following theorem and corollaries establish the coordinates of the vertices of a parallelogram, a rectangle, and a rhombus. They will be useful in proving some of the theorems in this section.

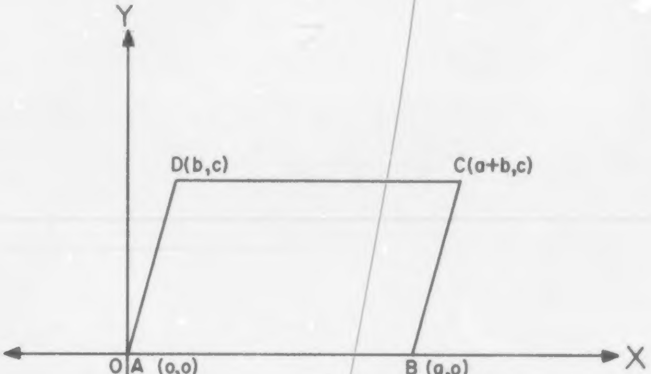
THEOREM 8-23. Given quadrilateral ABCD with $A = (0,0)$, $B = (a,0)$, $D = (b,c)$, then ABCD is a parallelogram if and only if $C = (a + b,c)$.

Proof: There are two things to prove:

(1) If ABCD is a parallelogram, then $C = (a + b,c)$.
(2) If $C = (a + b,c)$, then ABCD is a parallelogram.



(1) Suppose ABCD is a parallelogram. Let $C = (x,y)$. Since $\overleftrightarrow{AB}$ is horizontal, then $\overleftrightarrow{DC}$ is also horizontal. Therefore $y = c$. If $b \neq 0$ then neither $\overleftrightarrow{AD}$ nor $\overleftrightarrow{BC}$ is vertical. Since $\overline{AD} \parallel \overline{BC}$ it follows that $m_{\overline{AD}} = m_{\overline{BC}}$ and hence that $\frac{c}{b} = \frac{y}{x - a}$. But $y = c$. Therefore $x - a = b$, $x = a + b$, and $C = (a + b,c)$. If $b = 0$, then D is in the y-axis and $\overline{AD}$ is vertical. Since $\overline{BC} \parallel \overline{AD}$, $\overline{BC}$ is also vertical and $x = a$, $x = a + b$, and again $C = (a + b,c)$.

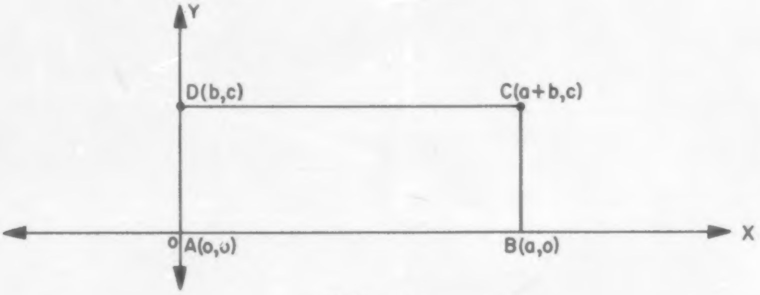


(2) If $C = (a + b,c)$ then $\overline{DC}$ is horizontal and hence parallel to $\overline{AB}$. Also $DC = |a + b - b| = |a|$, $AB = |a - 0| = |a|$. Then $DC = AB$ and $\overline{DC} \parallel \overline{AB}$. It follows that ABCD is a parallelogram.

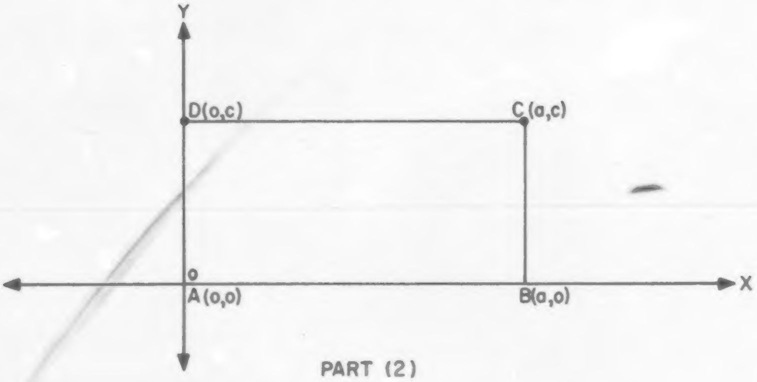
Corollary 8-23-1. If the coordinates of the vertices of a parallelogram are $A = (0,0)$, $B = (a,0)$, $C = (a + b,c)$, and $D = (b,c)$, then the parallelogram is a rectangle if and only if $b = 0$.

Proof: There are two things to prove:

(1) If ABCD is a rectangle, then $b = 0$.
(2) If $b = 0$, then ABCD is a rectangle.



PART (1)



PART (2)

(1) If ABCD is a rectangle, then $\angle A$ is a right angle and $\overline{AD} \perp \overline{AB}$. Therefore, D is in the y-axis and $b = 0$.

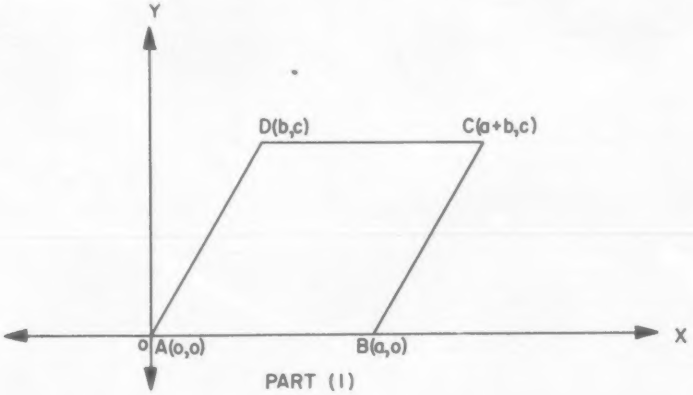
(2) If ABCD is a parallelogram and $b = 0$, then $D = (0,c)$. Since D is on the y-axis, we know that $\overline{AD} \perp \overline{AB}$ and $\angle A$ is a right angle. Therefore, ABCD is a rectangle.

Corollary 8-23-2. If the coordinates of the vertices of a parallelogram are $A = (0,0)$, $B = (a,0)$, $C = (a + b,c)$ and $D = (b,c)$ where $a > 0$, then the parallelogram is a rhombus if and only if $a = \sqrt{b^2 + c^2}$.

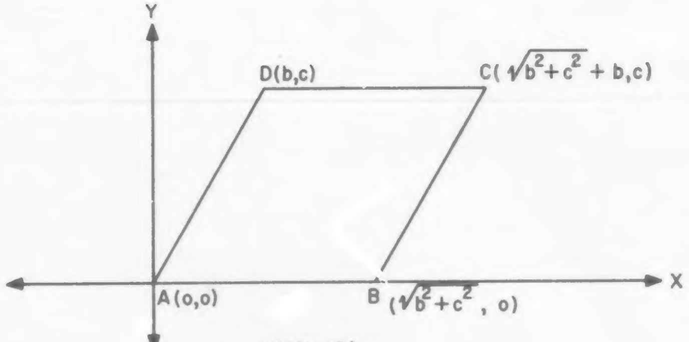
Proof: There are two things to prove:

(1) If ABCD is a rhombus, then $a = \sqrt{b^2 + c^2}$.

(2) If $a = \sqrt{b^2 + c^2}$, then ABCD is a rhombus.



PART (1)



PART (2)

(1) If ABCD is a rhombus, then by definition AB = AD . By the distance formula AB = a , and AD = $\sqrt{b^2 + c^2}$. By the substitution property of equality a = $\sqrt{b^2 + c^2}$.

(2) It is given that a = $\sqrt{b^2 + c^2}$. By the distance formula AB = a , and AD = $\sqrt{b^2 + c^2}$. By the substitution property of equality AB = AD . Since two adjacent sides of the parallelogram ABCD are congruent, the parallelogram is a rhombus.

We shall use the results of Theorem 8-23 and its corollaries to prove certain properties of the diagonals of a parallelogram, a rectangle, and a rhombus. The following experiment will help us to discover these relations.

Experiment

Draw several pictures of a parallelogram, a rectangle, a rhombus, and a square. Use a protractor and a ruler to discover the properties that appear to be true with respect to the diagonals of each of the given quadrilaterals. Record your findings in the chart by checking the quadrilateral which has the listed property.

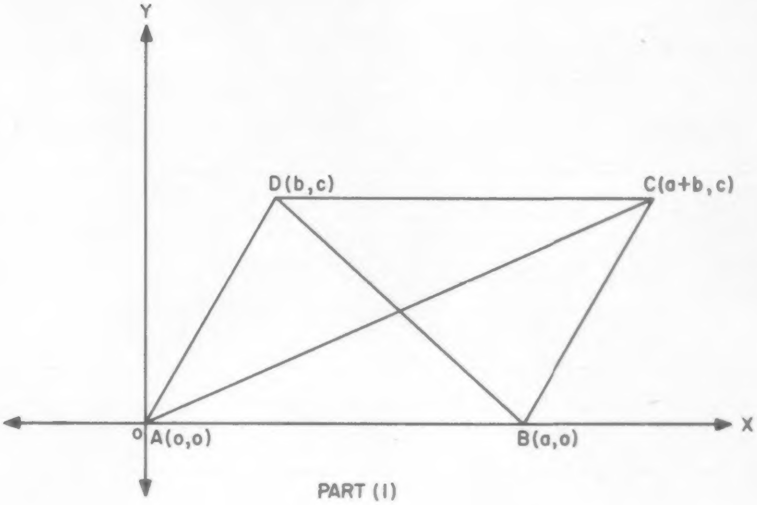
	Diagonals bisect each other	Diagonals are $\perp$	Diagonals bisect the angles
Parallelogram			
Rectangle			
Rhombus			
Square			

THEOREM 8-24. A quadrilateral is a parallelogram if and only if the diagonals bisect each other.

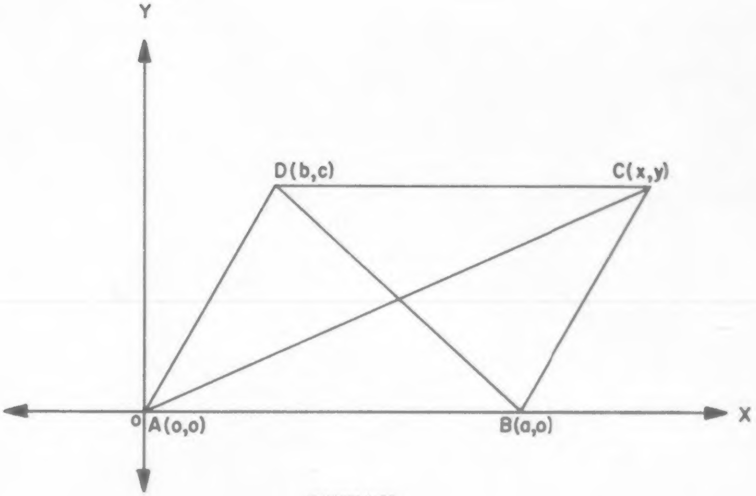
Proof: There are two things to prove:

(1) If ABCD is a parallelogram, then $\overline{AC}$ and $\overline{BD}$ bisect each other.

(2) If $\overline{AC}$ and $\overline{BD}$ bisect each other, then ABCD is a parallelogram.


Y
D(b,c)
C(a+b,c)
X
O
A(o,o)
B(a,o)

PART (1)


Y
D(b,c)
C(x,y)
X
O
A(o,o)
B(a,o)

PART (2)

(1) You will be asked to prove Part (1) of Theorem 8-24 in the next problem set.

(2) Let the coordinates of the quadrilateral be $A = (0,0)$, $B = (a,0)$, $C = (x,y)$, and $D = (b,c)$. Since $\overline{AC}$ and $\overline{BD}$ bisect each other they have the same midpoint. Thus

$$\left(\frac{x}{2}\, ,\, \frac{y}{2}\right) = \left(\frac{a+b}{2}\, ,\, \frac{c}{2}\right)$$

and $$\frac{x}{2} = \frac{a+b}{2}\ ;\ \frac{y}{2} = \frac{c}{2}\ .$$

From this we see that $x = a + b$ and $y = c$. Therefore $C = (a + b,c)$ and by Theorem 8-23 ABCD is a parallelogram.

You will be asked to prove the following theorems in the next problem set. You should note that there are two parts to the proof of each theorem. You should write out the two parts of the statement which must be proved before beginning your proof.

Theorem 8-23 and the two corollaries will help you set up the coordinate system for Theorems 8-25 and 8-26.

THEOREM 8-25. A parallelogram is a rectangle if and only if the diagonals are congruent.

THEOREM 8-26. A parallelogram is a rhombus if and only if the diagonals are perpendicular.

THEOREM 8-27. A parallelogram is a rhombus if and only if a diagonal bisects one of its angles.

Problem Set 8-13

1. Prove Part (1) of Theorem 8-24.
2. Prove Theorem 8-25.
3. Prove Theorem 8-26.
4. Prove Theorem 8-27.

5. List the properties of a rectangle that are not true of all parallelograms.

6. List the properties of a rhombus that are not true of all parallelograms.

7. Keeping in mind its definition, may a square be considered a rectangle? a rhombus? Then a square "inherits" the properties of ______ , ______ , and ______ .

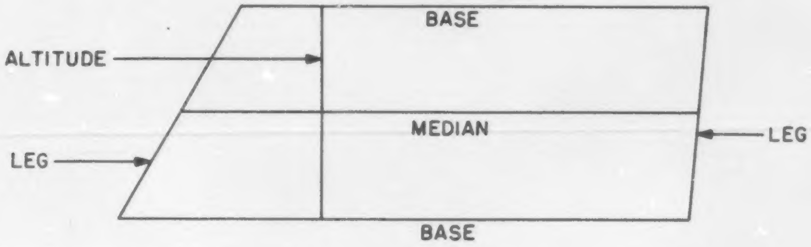
8. Make a chart like the following and check which figures have the listed properties.

	parallelogram	rectangle	rhombus	square
opposite sides are $\parallel$				
opposite sides are $\cong$				
opposite $\angle$s are $\cong$				
consecutive $\angle$s are supp.				
diagonals bisect each other				
diagonals are $\cong$				
diagonals are $\perp$				
diagonals bisect angles				
it is equilateral				
it is equiangular				
it is regular				

9. Starting with the set of all quadrilaterals explain how the set of parallelograms, rectangles, rhombuses and squares may be considered as subsets.

8-14. Trapezoids.

DEFINITION. A quadrilateral with one pair of sides parallel and the other pair of sides not parallel is called a trapezoid.



DEFINITION. The parallel sides of a trapezoid are called the bases of the trapezoid; the other two sides are called the legs of the trapezoid.

DEFINITION. If $\overline{AB}$ is a base of trapezoid ABCD then A and B are a pair of base angles of the trapezoid.

DEFINITION. A line segment which is perpendicular to the lines containing the bases of the trapezoid and which has its endpoints in these lines is called an altitude of the trapezoid.

DEFINITION. The line segment which connects the midpoints of the legs of a trapezoid is called the median of the trapezoid.

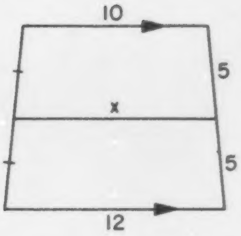
DEFINITION. A trapezoid whose legs are congruent is called an isosceles trapezoid.

Problem Set 8-14

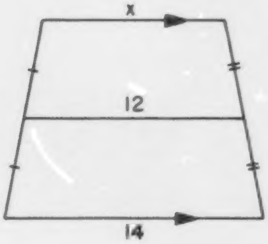
1. Prove that the median of a trapezoid is parallel to its bases and that its length is half the sum of the lengths of its bases. Hint: Let ABCD be the trapezoid with $A = (0,0)$, $B = (2a,0)$, $C = (2b,2c)$, $D = (2d,2c)$.

2. Using the result of Problem 1, find the length of the segment marked x or y in the following diagram. Parallel lines are indicated by arrows, lengths by numbers, and congruent segments by dashes.

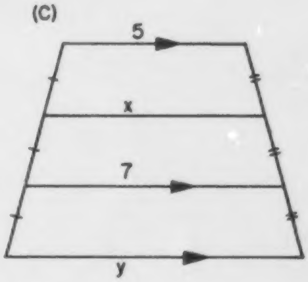
(a)



(b)



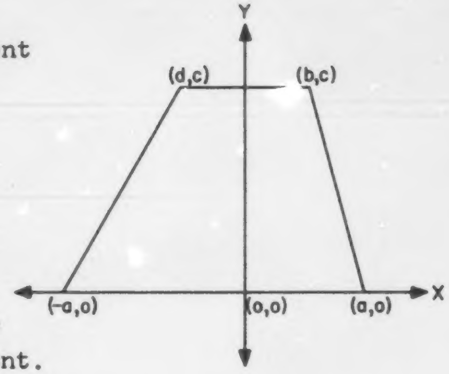
(c)



3. One angle of a trapezoid measures 100 . Can you find the measures of its remaining angles? If, in addition, you were told that the opposite angle has a measure of 70 , could you then find the measures of the two remaining angles? What are they?

4. Prove that a pair of base angles of a trapezoid are congruent if and only if the trapezoid is isosceles. (Decide first whether you will or will not use coordinates.)

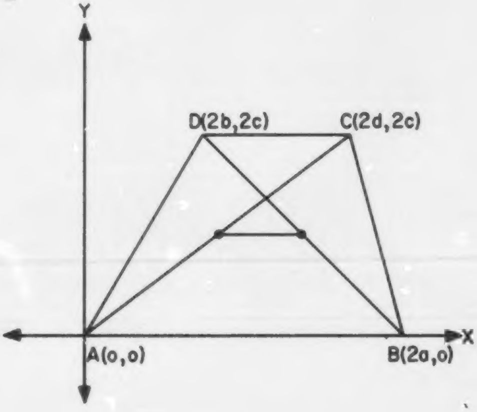
5. Prove that the diagonals of a trapezoid are congruent if and only if it is isosceles. (If you use coordinates you might choose coordinates as shown. Then you have to prove two statements.)



(1) If $d = -b$, then the diagonals are congruent.

*(2) If the diagonals are congruent, then $d = -b$.

6. Prove: The segment joining the midpoints of the diagonals of a trapezoid is parallel to the bases and equal in length to half the difference of their lengths.



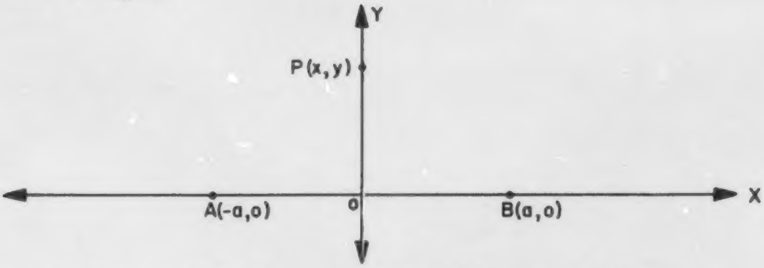
8-15. Concurrent Lines.

In this section we prove several statements which contain the phrase "the set of all points." Membership in the set is determined by a condition or a combination of conditions. The proof of such a statement consists of two parts. We must prove that:

1. Any point belonging to the set satisfies the given condition.
2. Any point that satisfies the given condition belongs to the set.

THEOREM 8-28. The set of all points in a plane which are equidistant from two given points in the plane is the perpendicular bisector of the segment joining the given points.

Proof: Given two points A and B and a plane which contains them. Choose $\overleftrightarrow{AB}$ as the x-axis and the midpoint of $\overline{AB}$ as the origin.



Then there is a real number a, $a \neq 0$, such that $A = (-a,0)$ and $B = (a,0)$. Then the y-axis is the perpendicular bisector of $\overline{AB}$. There are two parts to the proof.

(1) If P is in the y-axis, then $AP = PB$.

(2) If $AP = PB$, and P is in the xy-plane, then P is in the y-axis.

(1) If P is in the y-axis, then $P = (0,b)$ for some number b and

$$(AP)^2 = (-a - 0)^2 + (0 - b)^2 = a^2 + b^2 ,$$
$$(BP)^2 = (a - 0)^2 + (0 - b)^2 = a^2 + b^2 ,$$

and $AP = BP$.

(2) If $P(x,y)$ is any point such that $AP = PB$, then it follows from the distance formula that

$$(x + a)^2 + y^2 = (x - a)^2 + y^2$$
$$x^2 + 2ax + a^2 + y^2 = x^2 - 2ax + a^2 + y^2$$
$$4ax = 0 \text{ , and since } a \neq 0 ,$$
$$x = 0 .$$

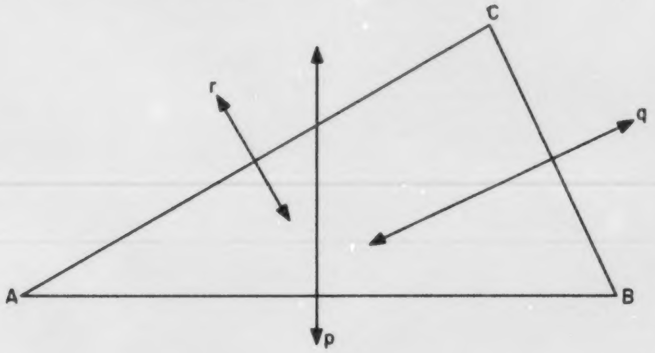
Therefore P is in the y-axis.

For the purposes of the next corollary it is convenient to have a definition of concurrent lines.

DEFINITION. The lines in a set of lines are called concurrent if and only if there is exactly one point which lies in all of them; the segments in a set of segments are called concurrent if and only if there is exactly one point which lies in all of them.

According to this definition and our earlier definitions, we note that concurrent rays lie on concurrent lines, or in the special case of two opposite rays, they lie on the same line.

Corollary 8-28-1. The perpendicular bisectors of the sides of a triangle are concurrent at a point equidistant from the vertices of the triangle.

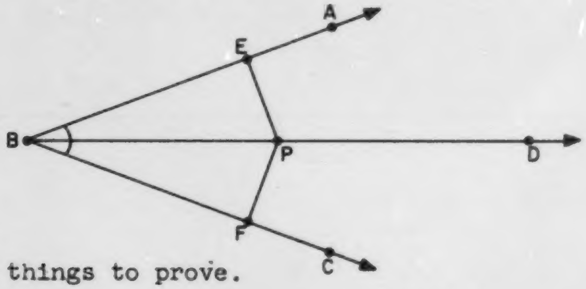


Given triangle ABC . Let p, q and r be the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$, respectively. Either p and r intersect or p is parallel to r . If we assume $p \parallel r$ then $AB \perp r$. But $r \perp AC$ by hypothesis. What theorem does this contradict? The assumption that p is parallel to r is false. Therefore p intersects r at a point.

If O is the point of intersection of p and r , then OB = OA and OA = OC by Part (1) of Theorem 8-28. Therefore, OB = OA = OC by the transitive property of equality. Therefore, O is in q by Part (2) of Theorem 8-28. This proves that p, q, and r are concurrent at a point equidistant from A, B, and C, the vertices of the triangle.

THEOREM 8-29. The set of all points in the interior of an angle which are equidistant from the lines which contain the sides of the angle is the interior of the midray of the angle.

Proof: We construct a proof without coordinates. Let an angle ABC and its midray $\overrightarrow{BD}$ be given.



We have two things to prove.

(1) If P is in the interior of $\overrightarrow{BD}$, then the distance from P to $\overleftrightarrow{AB}$ equals the distance from P to $\overleftrightarrow{BC}$.

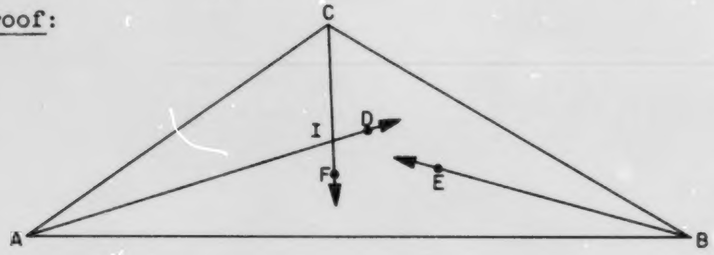
(2) If P is in the interior of $\angle ABC$ and if the distance from P to $\overleftrightarrow{BA}$ equals the distance from P to $\overleftrightarrow{BC}$, then P is in the interior of $\overrightarrow{BD}$.

(1) Suppose P is an interior point of $\overrightarrow{BD}$. Since $m\angle ABC < 180$, then $m\angle PBC < 90$ and it follows that the foot of the perpendicular from P to $\overleftrightarrow{BC}$ is some point on $\overrightarrow{BC}$, call it F. Similarly, the foot of the perpendicular from P to $\overleftrightarrow{BA}$ is some point in [illegible], call it E. $\triangle BPE \cong \triangle BPF$ by S.A.A. and $PE = PF$. Hence the distances of P from $\overleftrightarrow{BA}$ and $\overleftrightarrow{BC}$ are equal.

(2) Since P is in the interior of the angle and $\overline{PE} \perp \overleftrightarrow{BA}$, $\overline{PF} \perp \overleftrightarrow{BC}$, and $PE = PF$, then $\triangle PBE \cong \triangle PBF$ and $m\angle FBP = m\angle EBP$. Therefore $\overrightarrow{BP}$ is the midray of $\angle ABC$.

Corollary 8-29-1. The lines which contain the angle bisectors of the angles of a triangle are concurrent at a point equidistant from the sides of the triangle.

Proof:



Let $\triangle ABC$ with angle bisectors $\overrightarrow{AD}$, $\overrightarrow{BE}$, $\overrightarrow{CF}$ be given. Now $\overrightarrow{AD}$ and $\overrightarrow{CF}$ (except for the points A and C) lie in the same halfplane with edge $\overrightarrow{AC}$. Also $\overrightarrow{AD}$ and $\overrightarrow{CF}$ are not parallel (since the measures of $\angle CAD$ and $\angle FCA$ are each less than 90). Let I be their point of intersection. From Part (1) of Theorem 8-25 it follows that I is equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ and also equidistant from $\overleftrightarrow{AC}$ and $\overleftrightarrow{CB}$. It follows that I is equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ and by Part (2) of Theorem 8-25 that I lies in $\overrightarrow{BE}$. This means that $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ are concurrent in the point I , and I is equidistant from $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{CB}$.

Problem Set 8-15

1. Given A(-3,0) , B(0,4) , C(5,0) . Plot points A, B and C and show by a drawing how to locate a point D such that DA = DB = DC . Explain your drawing and state the theorem (or corollary) that suggested it.
2. Given A(-2,0) , B(0,-6) , and C(3,0). Using a protractor find a point D such that the distances from D to $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{CA}$ are equal. State the theorem (or corollary) that suggested your drawing.
3. Given A(-3,0) , B(5,0) , and C(0,4). Use a ruler and protractor to:

   (a) Find a point X such that AX = BX and the distances from X to $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ are equal.

(b) Find a point on the x-axis that is equally distant from A and B .

(c) Find a point on the y-axis that is equally distant from $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$.

4. In triangle ABC , D is the midpoint of $\overline{BC}$, E the midpoint of $\overline{CA}$, and F the midpoint of $\overline{AB}$.

(a) If $AB = 12$, $CB = 9$, and $AC = 10$, find DE, EF, FD .

(b) Prove that $\overline{DE} \parallel \overline{AB}$; that $\overline{EF} \parallel \overline{BC}$; that $\overline{FD} \parallel \overline{CA}$.

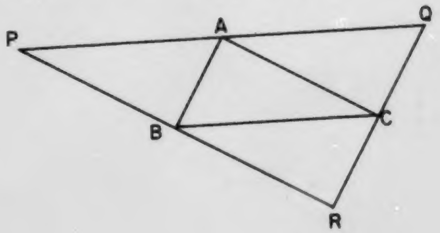
(c) The perpendicular bisector of $\overline{AB}$ is also perpendicular to __?__ . The perpendicular bisector of BC is also perpendicular to __?__ . The perpendicular bisector of $\overline{CA}$ is also perpendicular to __?__ .

(d) Are the lines that contain the altitudes of triangle DEF concurrent? Explain.

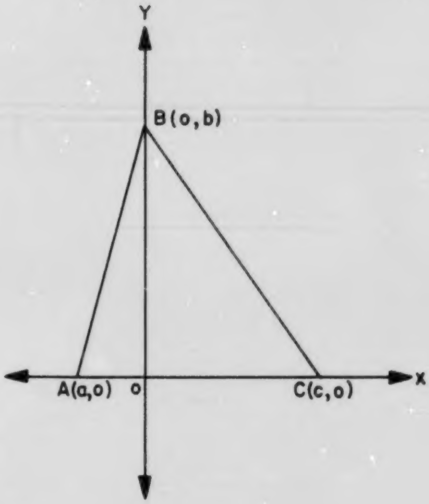
5. We sketch two proofs of the following statement: The lines that contain the altitudes of a triangle are concurrent. You are to fill in the missing parts of each proof. Then decide which proof seems to be more satisfying.

Proof I:

Let ABC be the triangle. Consider the line through A parallel to $\overline{BC}$; the line through B parallel to $\overline{AC}$; the line through C parallel to $\overline{AB}$. Let these lines meet in P, Q, R as shown in the diagram. Show that $\triangle PAB \cong \triangle CBA$ and that $PA = CB$. Similarly show that $CB = AQ$. It follows then that $PA = AQ$. In the same way $PB = BR$ and $RC = CQ$. The altitude from A to $\overline{BC}$ is contained in the perpendicular bisector of $\overline{PQ}$. Complete the proof.



Proof II:



Let the triangle be ABC and choose axes so that $A = (a,0)$, $B = (0,b)$, and $C = (c,0)$. Then the y-axis contains the altitude from B to $\overline{AC}$.

$m_{\overline{CB}} = -\frac{b}{c}$, $m_{\overline{AB}} = -\frac{b}{a}$.

Why? Therefore the slope of the altitude, h_a from A is $\frac{c}{b}$ and the slope of the altitude from C, h_c, is $\frac{a}{b}$. Why?

The line that contains h_a is

$\{(x,y) : x = a + bk, \ y = 0 + ck, \ k \text{ is real}\}$;

the line that contains h_c is

$\{(x,y) : x = c + bp, \ y = 0 + ap, \ p \text{ is real}\}$.

Setting $k = -\frac{a}{b}$ we find that $(0, -\frac{ac}{b})$ is contained in h_a. Why? Setting $p = -\frac{c}{b}$ we find that $(0, -\frac{ac}{b})$ is also contained in h_c. Why? Since the x-coordinate of this point is 0, the point is on the y-axis, which contains h_b. Therefore the lines which contain the altitudes are concurrent.

6. Prove Corollary 8-28-1 by coordinates.

8-16. Summary.

In this chapter we defined coordinates in a plane and we used them as a tool in formal geometry. We have seen some "neat" proofs involving coordinates. In other situations we have decided to write proofs without coordinates. In constructing a proof using coordinates it is usually wise to set up a coordinate system which makes the expressions involving coordinates as simple as possible.

We developed several expressions for the coordinates of the points of a line, with considerable emphasis on the use of set-builder notation and parametric equations. We defined the slope of a non-vertical line and used it to get conditions for perpendicularity and parallelism of oblique lines:

$$p \perp q \quad \text{if and only if} \quad m_p \cdot m_q = -1 ,$$

$$p \parallel q \quad \text{if and only if} \quad m_p = m_q .$$

We developed several equations for lines:

the two-point form,
the point-slope form.

We developed several formulas:

the distance formula,
the midpoint formula.

The chapter includes several theorems on triangles:

one about a line joining midpoints of two sides,
one about concurrence of angle bisectors,
one about concurrence of perpendicular bisectors of sides.

The following table summarizes several definitions and theorems which are concerned with quadrilaterals. Each line in the table yields a statement of the form: An A is a B if and only if C . Proofs for statements with no reference listed are easy.

A	B	C	by
quadrilateral	parallelogram	opposite sides are parallel	Defn. Ch. 6
quadrilateral	parallelogram	opposite sides are congruent	Ch. 8
quadrilateral	parallelogram	two sides are parallel and congruent	Ch. 6
quadrilateral	parallelogram	diagonals bisect each other	Th.8-24
quadrilateral	rhombus	all sides are congruent	
quadrilateral	square	all sides are congruent and all angles are right angles	
quadrilateral	trapezoid	exactly one pair of sides is parallel.	Defn.
quadrilateral	rectangle	all angles are congruent	Th.8-20
parallelogram	rhombus	all sides are congruent	Th.8-21
parallelogram	rectangle	diagonals are congruent	Th.8-25
parallelogram	rectangle	all angles are right angles	
parallelogram	rhombus	diagonals are perpendicular	Th.8-26
parallelogram	rhombus	diagonal bisects one angle	Th.8-27
rectangle	square	all sides are congruent	
rhombus	square	all angles are congruent	

Review Problems

1. Plot the graph of each of the following.

(a) $p = \{(x,y) : x = 2 , 0 < y < 3\}$.

(b) $q = \{(x,y) : y = 2 , 0 \leq x \leq 3\}$.

(c) $r = \{(x,y) : x = 1 + 2k , y = 1 - 2k ,$ k is an integer and $-2 < k < 2\}$.

(d) $s = \{(x,y) : x + y = 3 , 0 < x < 3$ and $0 < y < 3\}$.

(e) $t = \{(x,y) : x + y = 3 , -3 < x < 0\}$.

2. Is the following information sufficient to prove quadrilateral ABCD a parallelogram?

(a) $AB = DC$; $\overline{AB} \parallel \overline{DC}$.
(b) $AB = DC$; $AD = BC$.
(c) $\overline{AB} \parallel \overline{DC}$; $\overline{AD} \parallel \overline{BC}$.
(d) $AB = DC$; $\overline{AD} \parallel \overline{BC}$.
(e) $\angle A \cong \angle C$; $\angle B \cong \angle D$.
(f) $\overline{AC}$ bisects $\overline{BD}$.
(g) $\overline{AC}$ bisects $\overline{BD}$; $\overline{BD}$ bisects $\overline{AC}$.
(h) $\overline{AB} \perp \overline{AD}$; $\overline{DC} \perp \overline{AD}$; $\overline{BC} \perp \overline{CD}$.
(i) $\triangle ABD \cong \triangle CBD$.
(j) $\triangle ABD \cong \triangle CDB$.
(k) $\angle A \cong \angle C$; $\overline{AB} \parallel \overline{DC}$.
(l) $\angle A \cong \angle C$; $AB = DC$.

3. The diagonals of a rhombus are 16 and 30 . Find the perimeter of the rhombus.

4. The ratio of the lengths of two sides of a rectangle is 3 : 4 . The length of the diagonal of the rectangle is 40 ; find the lengths of the sides of the rectangle.

5. Three vertices of rectangle ABCD are $A(-1,-1)$, $B(3,-1)$, and $C(3,5)$.

(a) What is the fourth vertex?
(b) What is the midpoint of $\overline{AB}$?
(c) What is the midpoint of $\overline{AC}$?
(d) What is AB ?
(e) What is AC ? Show that $AC = BD$.
(f) Write $\overleftrightarrow{AB}$ using parametric equations.
(g) Write $\overleftrightarrow{AC}$ using parametric equations.
(h) Find Q on $\overrightarrow{AC}$ such that $AQ = 4AC$.
(i) Write parametric equations for the line through C that is perpendicular to $\overline{AC}$.

6. An isosceles triangle has vertices $(0,0)$, $(4a,0)$, $(2a,2b)$.

(a) What is the slope of the median from $(0,0)$, if any?
(b) What is the slope of the median from $(4a,0)$, if any?
(c) Find the slope of the median from $(2a,2b)$, if any.

7. In square $ABCD$, R is the midpoint of $\overline{BC}$ and S is the midpoint of $\overline{CD}$. $\overline{AR}$ intersects $\overline{BS}$ in T .

   (a) Prove that $BS = AR$.
   (b) Prove that $\overline{BS} \perp \overline{AR}$.
   *(c) Prove that $TD = AB$.

   Hint: Let $A = (0,0)$ and $B = (2a,0)$.

8. Prove that the median of a trapezoid bisects a diagonal.

9. (a) What is an equation of the x-axis?
   (b) What is an equation of the y-axis?
   (c) Show that all points of both axes satisfy the equation $xy = 0$.

10. A rhombus $ABCD$ has A at the origin and $\overline{AB}$ in the positive x-axis, $m\angle A = 60$, $AB = 6$, C is in Quadrant I.

    (a) What are the coordinates of C ?
    (b) What are the coordinates of D ?
    (c) Find AC .
    (d) Show that $AC = \sqrt{3}\,BD$.
    (e) Using parametric equations express $\overrightarrow{AC}$.

11. Write an equation for the set of points

    (a) whose distances to $(-3,0)$ and $(5,0)$ are equal.
    (b) whose distances to the x-axis is 3 .
    (c) whose distances to the x- and y-axes are equal.
    (d) whose distances to the horizontal lines $y = -2$ and $y = 8$ are equal.
    (e) whose x-coordinates are 12 .
    (f) whose y-coordinates are -8 .

12. Show that triangle ABC is a right isosceles triangle if $A = (3,4)$, $B = (-1,5)$, $C = (-2,1)$.

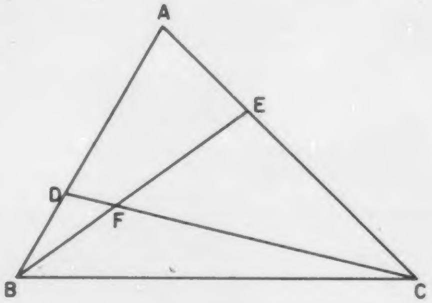
13. Using parametric equations express the set of points equally distant from $A(0,4)$ and $B(-8,0)$.

14. The point $A(c,6)$ is equally distant from $B(1,1)$ and $C(3,5)$. Find the value of c .

15. The distance from $(h,3)$ to the x-axis is twice its distance to the y-axis. Find h. (Two answers.)

16. ABCD is a parallelogram. Show that the segment that joins D to the midpoint of $\overline{AB}$ trisects $\overline{AC}$.

17. In triangle ABC, D is in $\overline{AB}$, and E is in $\overline{AC}$. $AD = 2DB$ and $CE = 2EA$. $\overline{BE}$ and $\overline{DC}$ intersect in F. Show that $\frac{BF}{FE} = \frac{3}{4}$ and $\frac{DF}{FC} = \frac{1}{6}$.



Chapter 9

PERPENDICULARITY, PARALLELISM, AND COORDINATES IN SPACE

9-1. Introduction.

Our first contact with points, lines, and planes in space was in Chapter 2, but since then our work has been almost completely restricted to points and lines in a single plane. Now, having investigated plane geometry in some detail, we are ready to turn our attention to space geometry. In particular, in this chapter we extend the ideas of perpendicularity and parallelism to figures which may not be contained in a plane.

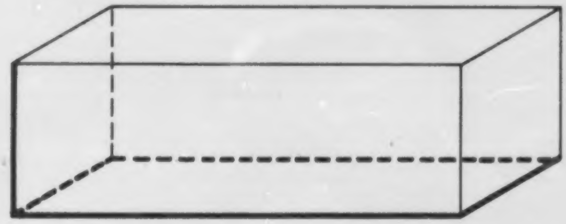
Most of the results we are going to discuss are familiar to us from our past experience. However, we often miss the essential features of things we have seen a hundred times, and certain results which are true in the plane are not true in space. Moreover, without practice it is hard to visualize geometric relations in space and harder still to represent them by drawings on a sheet of paper. To save time, it therefore seems wise to omit the proofs of most of our theorems and concentrate instead on getting a thorough understanding of the results themselves. Fortunately, the proofs of the theorems in this chapter are quite similar to the deductive arguments we have seen in previous chapters, and a few samples will be an adequate indication of how the rest can be constructed. Of course at any time you are free to use any theorem that has been previously stated, whether it has been proved in the text or not.

In preparation for the work which follows, it will be helpful for you to review the simple space relations introduced in Sections 2-5, 2-6, and 2-7, and then to go carefully through the exploratory problems which are given below. The ability to make and interpret drawings of three-dimensional configurations will be of great value to you through the rest of this course.

Be sure that you can do these two things. Appendix V offers many suggestions which may be helpful to you.

Exploratory Problems

1. In the following sketch of a rectangular block, certain combinations of edges, considered separately, suggest certain configurations of lines and planes.



In each of the following, copy the drawing of the block and darken the appropriate edges to suggest your idea of the indicated configuration.

(a) Two distinct intersecting lines.
(b) Two distinct parallel lines.
(c) Two lines which are neither intersecting nor parallel.
(d) Three mutually perpendicular lines.
(e) Three parallel lines which are not coplanar.
(f) A line intersecting one of two parallel lines but not the other.
(g) Two distinct lines which are perpendicular to the same line and parallel to each other.
(h) Two lines which are perpendicular to the same line at different points but are not parallel to each other.
(i) A line parallel to a plane.
(j) Two distinct lines which are parallel to the same plane and parallel to each other.
(k) Two lines which are parallel to the same plane but not parallel to each other.
(l) Two distinct parallel planes.

(m) Two perpendicular planes.
(n) Three mutually perpendicular planes.
(o) A plane perpendicular to each of two distinct parallel planes.
(p) Two distinct lines perpendicular to the same plane.
(q) Two distinct planes perpendicular to the same line.

2. Without including any unnecessary lines, make drawings of your own to suggest the configurations in Parts (d), (h), (i), (j), (k), (l), (m), (n), (o), (p), (q) of the preceding problem.

9-2. Perpendicularity Relations.

In Section 4-8 we defined what we mean by two perpendicular lines. In this section we are going to discuss the perpendicularity relation between a line and a plane. Before we define this formally, however, you should study the following experiments and from the clues they give you, you should try to make up a definition of your own.

Experiments

1. In Section 4-10 we learned that in a plane there is a unique line which is perpendicular to a given line at a given point. Is this true in space? Hold two pencils so that they appear perpendicular to each other. Can you hold one of the pencils in a different position and still have it appear perpendicular to the other at the same point? How many different positions can one pencil assume and remain perpendicular to the second pencil at the same point? Do you think these "perpendiculars" might lie in the same plane? Would such a plane be perpendicular to the other pencil?

2. Place a sheet of paper on your desk. Hold your pencil so that it appears perpendicular to the paper. With a second pencil, draw a line on the page that appears to be perpendicular to the first pencil.

(a) Can you shift your first pencil so that it remains perpendicular to the line at the same point but is not perpendicular to the paper?

(b) Draw another line, intersecting the first. Now, place your pencil so that it appears perpendicular to both lines at their point of intersection. Does the pencil appear to be perpendicular to the plane of the paper? Can you hold your pencil so that it is perpendicular to the paper?

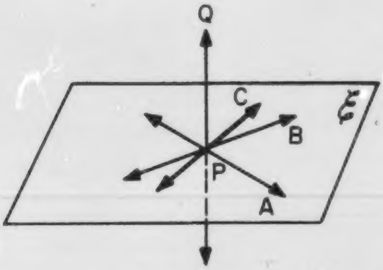
(c) Draw additional lines through the point of intersection. Does the pencil appear to be perpendicular to each of them at that point? Is it still perpendicular to the plane?

(d) What do you think would be a good definition of a line perpendicular to a plane?

The preceding experiments lead us to the following definition:

> DEFINITION. A line and a plane are perpendicular to each other if and only if they intersect and every line lying in the plane and passing through the point of intersection is perpendicular to the given line.

The following figure suggests the relations described by this definition:



$\overleftrightarrow{PA}$, $\overleftrightarrow{PB}$, $\overleftrightarrow{PC}$, ... all lying in plane ξ are perpendicular to $\overleftrightarrow{PQ}$.

The results which we obtain in this chapter can all be derived without additional postulates. However, our development proceeds much more easily if we accept as postulates two theorems whose proofs are long and rather involved.

The first, Postulate 24, should remind us of Theorems 4-21 and 5-11 which deal with the existence of a line containing a given point and perpendicular to a given line. This postulate is all that we need at present. The second, Postulate 25, we shall introduce in Section 9-4.

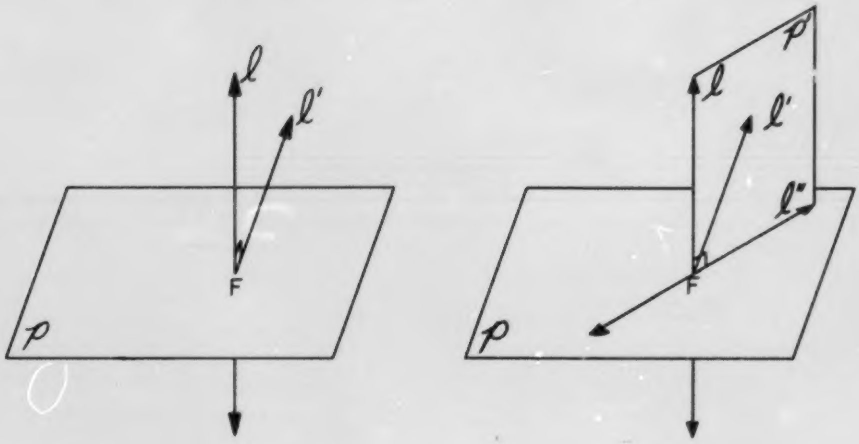
> Postulate 24. There is a unique plane which contains a given point and is perpendicular to a given line.

We should understand clearly that in this postulate no restriction is placed on the given point. It can equally well be a point on the given line or a point which is not on the given line. The postulate says simply that wherever the point may be, there is always one and only one plane which contains the point and is perpendicular to the given line.

From the definition of perpendicularity, we know that if a plane $\mathcal{P}$ is perpendicular to a line ℓ at a point F , on ℓ , then every line in $\mathcal{P}$ which passes through F is perpendicular to ℓ . However, we do not yet know whether there can also be lines perpendicular to ℓ at F which do not lie in $\mathcal{P}$. The following theorem answers this question for us.

THEOREM 9-1. The plane which is perpendicular to a given line at a given point contains every line which is perpendicular to the given line at that point.

Proof: Let ℓ be any line and let $\mathcal{P}$ be the plane which is perpendicular to ℓ at the point F . What we must show is that if ℓ' is any line perpendicular to ℓ at F , then ℓ' lies in $\mathcal{P}$.



Now the intersecting lines ℓ and ℓ' determine a plane, say $\mathcal{P}'$, and by Postulate 9 this plane intersects the plane $\mathcal{P}$ in a line, say ℓ'' .

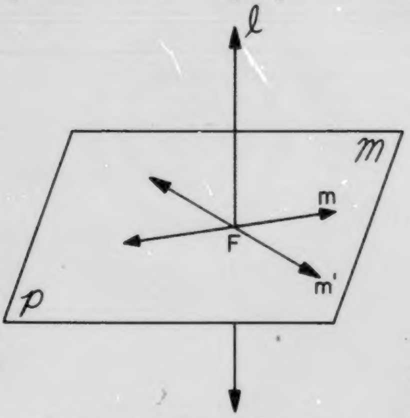
Moreover, since ℓ'' lies in the perpendicular plane $\mathcal{P}$, it must be perpendicular to ℓ at F . Hence, in the plane $\mathcal{P}'$ both ℓ' and ℓ'' are perpendicular to ℓ at the point F .

But by Theorem 4-21, in a given plane there is exactly one perpendicular to a given line at a given point. Hence ℓ' and ℓ'' must be the same line. That is, ℓ' must lie in the perpendicular plane $\mathcal{P}$, as asserted.

According to our definition, before we can say that a line ℓ is perpendicular to a plane $\mathcal{P}$ at a point F , we must be sure that ℓ is perpendicular to every line in $\mathcal{P}$ which passes through F . The next theorem tells us that we do not need nearly this much information to be sure that a line is perpendicular to a plane.

THEOREM 9-2. If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane determined by the two lines.

Proof: Let m and m' be two distinct lines which intersect at the point F and let ℓ be a line which is perpendicular to both m and m' at F . Let $\mathcal{M}$ be the plane determined by the intersecting lines, m and m' , and let $\mathcal{P}$ be the plane which is perpendicular to ℓ at F .

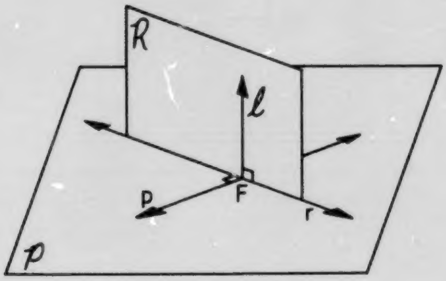


According [illegible] the last theorem both m and m' must lie in $\mathcal{P}$. Hence the planes $\mathcal{M}$ and $\mathcal{P}$ have both m and m' in common. Therefore, by Theorem 2-10, $\mathcal{M}$ and $\mathcal{P}$ must be the same plane; that is, the plane determined by the two lines, m and m' , is the plane which is perpendicular to ℓ at F , as asserted.

Postulate 24 assures us that there is a unique plane which is perpendicular to a given line at a given point, but it does not answer the corresponding question of the existence of a line which is perpendicular to a given plane at a given point. However, this is settled by the following theorem.

THEOREM 9-3. There is a unique line which is perpendicular to a given plane at a given point in the plane.

We shall omit the proof of this theorem. The general outline is suggested by the following figure. $\mathcal{R}$ is the plane which is perpendicular at the given point, F , to any particular line, p , which lies in the given plane, $\mathcal{P}$, and passes through F . The required perpendicular ℓ is the line in $\mathcal{R}$ which is perpendicular at F to the line, r , in which $\mathcal{P}$ and $\mathcal{R}$ intersect.

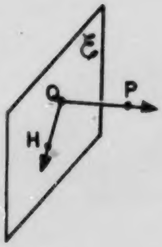


The corresponding theorem dealing with the existence of a line which passes through a given point not in a plane and is perpendicular to the plane is more conveniently handled a little later after we have discussed parallel relations in space.

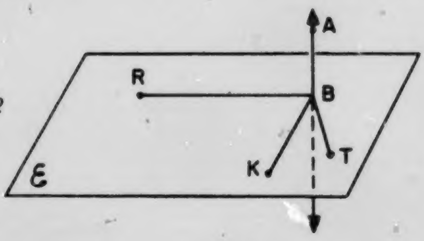
Problem Set 9-2

In each of the following problems, draw your own diagram as part of the proof.

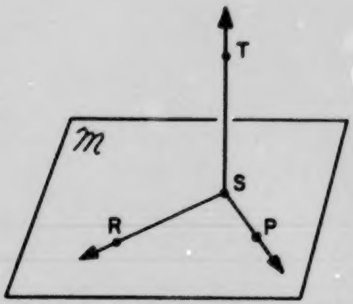
1. In the figure, if $\angle PQH$ is a right angle and Q and H are in $\mathcal{E}$, should you infer from the definition of a line perpendicular to a plane that $\overleftrightarrow{PQ} \perp \mathcal{E}$? Justify your answer.



2. In the figure, points B, R, K and T are in plane $\mathcal{E}$, and $\overleftrightarrow{AB} \perp \mathcal{E}$. Which of the following angles must be right angles: $\angle ABR$, $\angle ABK$, $\angle RBT$, $\angle TBA$, $\angle KBR$? Why?



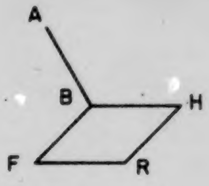
3. In the figure, plane m contains the noncollinear points R, S, P, but m does not contain T .



   (a) Do points R, S, and T determine a plane? Why?

   (b) If $\overrightarrow{SP}$ is perpendicular to the plane of R, S, T , which angles in the figure must be right angles? Why?

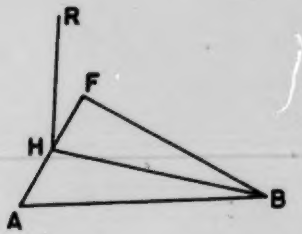
4. In the figure, the point A and the square FRHB are not coplanar; $\overline{AB} \perp \overline{FB}$.



   (a) How many planes are determined by pairs of segments in the figure? Name them.

   (b) At least one of the segments in this figure is perpendicular to one of the planes asked for in Part (a). Which segment? Which plane? Justify your answer.

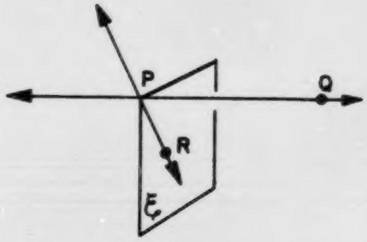
5. In the figure, point R and triangle ABF are not coplanar, $\triangle ABF$ is isosceles with vertex B , H is the midpoint of $\overline{AF}$, and $\overline{RH} \perp \overline{HB}$.



   (a) How many different planes are determined by pairs of segments in the figure? Name them.

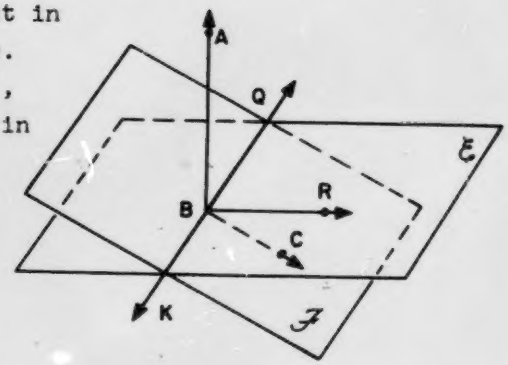
   (b) Find a segment that is perpendicular to a plane. State the perpendicularity and the theorems which justify your statement.

6. In the figure, $\overleftrightarrow{QP} \perp \mathcal{E}$ at P and $\overleftrightarrow{QP} \perp \overleftrightarrow{PR}$. Must $\overleftrightarrow{PR}$ lie in plane $\mathcal{E}$? Why?

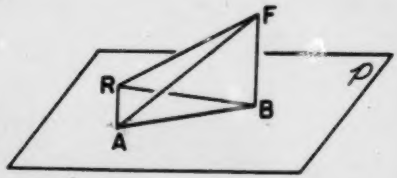


7. Planes $\mathcal{E}$ and $\mathcal{F}$ intersect in $\overleftrightarrow{KQ}$, as shown in the figure. $\overleftrightarrow{AB} \perp \mathcal{E}$, $\overleftrightarrow{BR}$ lies in $\mathcal{E}$, plane ABR intersects $\mathcal{F}$ in $\overleftrightarrow{BC}$.

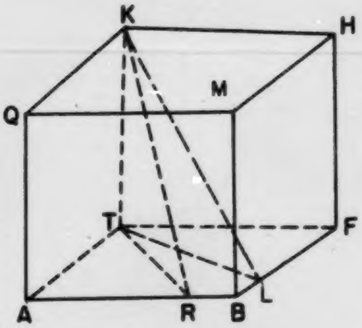
   (a) Is $\overleftrightarrow{AB} \perp \overleftrightarrow{BR}$? Why?
   (b) Is $\overleftrightarrow{AB} \perp \overleftrightarrow{KQ}$? Why?
   (c) Is $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$? Why?



8. In this figure, $\overline{FB} \perp$ plane $\mathcal{P}$, and in $\triangle RAB$, which lies in plane $\mathcal{P}$, $BR = BA$. Prove $\triangle ABF \cong \triangle RBF$ and $\angle FAR \cong \angle FRA$.



9. Given the cube shown, with $BR = BL$. Does $KR = KL$? Prove that your answer is correct.



(Since we have not yet given a precise definition of a cube, we state here, for use in your proof, the essential properties of the edges of a cube:

The edges of a cube consist of twelve congruent segments, related as shown in the picture, such that any two intersecting segments are perpendicular.)

9-3. Parallel Relations.

In this section we are going to investigate parallel relations between lines and planes in space, and this requires that we first define what we mean by saying that a line and a plane, or two planes, are parallel. The following definitions are natural extensions of the definition of parallel lines which we gave in Section 6-2.

DEFINITION. A line and a plane whose intersection does not consist of exactly one point are *parallel to* each other.

DEFINITIONS. Two planes (whether distinct or not) whose intersection is not a line are *parallel planes*, and each is *parallel to* the other.

With these definitions in mind, the following experiments should help you to visualize the properties we are going to discuss.

Experiments

1. Draw a line on a sheet of paper on your desk. Now hold two pencils above your desk so that each appears parallel to the line. Do the pencils appear parallel to each other? Can you hold them so they are parallel to the line and not to each other?

2. (a) If two distinct lines are parallel to a plane, are they necessarily parallel to each other? The Parallel Postulate tells us that there is a unique line which contains a given point and is parallel to a given line. Do you think there is a unique line which contains a given point and is parallel to a given plane? Hold two pencils so that they "intersect." Can you hold them so that both of them are also parallel to the desk top?

   (b) Hold the two pencils so that they represent skew (noncoplanar) lines. Can you hold them so that they are both parallel to the desk top?

3. (a) We have learned that, in a plane, if a line intersects one of two parallel lines in a point, it intersects the other in a point also. Is this true in space? Draw two distinct parallel lines on a sheet of paper. Can you hold a pencil so it will intersect one of the parallel lines but not the other?

   (b) Suppose a plane intersects one of two parallel lines in a point. Do you think it must intersect the other also?

   (c) Suppose a line intersects one of two parallel planes in a point. Do you think it must intersect the other plane also?

   (d) Sketch diagrams to illustrate Parts (a), (b), (c).

4. Suppose a plane intersects one of two parallel planes. Do you think it must intersect the other plane also? If a plane intersects each of two parallel planes, what can you say about the lines of intersection? In your classroom consider the parallel walls "intersected" by the floor. Are the lines of intersection parallel? Think of a bookcase. The shelves are parallel planes, the end panel an intersecting plane. What about the lines of intersection? Draw a diagram of two distinct parallel planes intersected by a third plane.

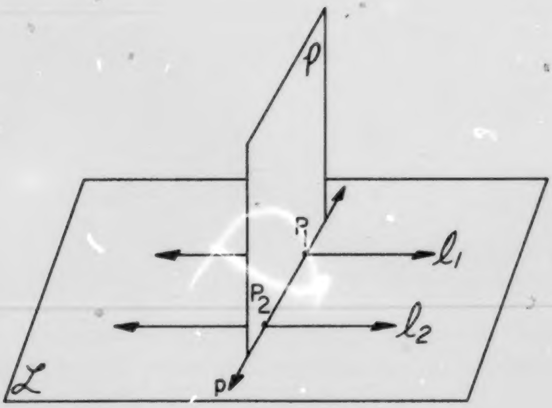
As Experiment 3 (a) in the preceding list suggests, in space geometry it is not true that if a third line meets one of two parallel lines it must meet the other one also. However there are analagous theorems which are true in space, and to these we now turn our attention.

THEOREM 9-4. If a plane intersects one of two distinct parallel lines in a point, it intersects the other line in a point also.

Proof: Let ℓ_1 and ℓ_2 be two distinct parallel lines, contained in a plane $\mathcal{L}$, and let $\mathcal{P}$ be a plane which intersects one of the lines, say ℓ_1 , in a single point, P_1 .

Clearly, $\mathcal{P}$ cannot contain ℓ_2 because otherwise, by Theorem 2-9, it would coincide with $\mathcal{L}$, and hence contain ℓ_1 , contrary to the hypothesis that it meets ℓ_1 in just one point. Therefore $\mathcal{P}$ can have at most one point in common with ℓ_2 .

Now by Postulate 9, since $\mathcal{L}$ and $\mathcal{P}$ have a point, P_1 , in common,



they must have a line, say p , in common. Moreover, since $\mathcal{P}$ meets each of ℓ_1 and ℓ_2 in at most one point, p must be distinct from both ℓ_1 and ℓ_2 .

Now in the plane $\mathcal{L}$, the line p meets one of the two parallel lines, ℓ_1 and ℓ_2 , in a single point, P_1 . Hence it must also meet the other line ℓ_2 , in a point, say P_2 .

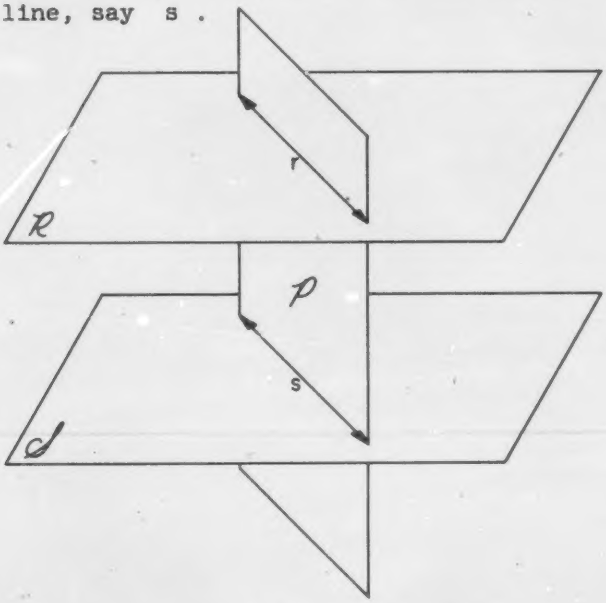
Since the line p is contained in the plane $\mathcal{P}$, the point P_2 is also contained in $\mathcal{P}$. Therefore $\mathcal{P}$ intersects ℓ_2 in a single point, as asserted.

The next theorem follows easily from the preceding one, and we shall leave its proof as a problem.

THEOREM 9-5. If a plane is parallel to one of two parallel lines, it is also parallel to the other.

THEOREM 9-6. If a plane intersects each of two distinct parallel planes, the intersections are two distinct parallel lines.

Proof: Let $\mathcal{R}$ and $\mathcal{S}$ be two distinct parallel planes and let $\mathcal{P}$ be a plane which intersects both $\mathcal{R}$ and $\mathcal{S}$. Since $\mathcal{R}$ and $\mathcal{S}$ do not intersect, $\mathcal{P}$ must be distinct from both $\mathcal{R}$ and $\mathcal{S}$. By Postulate 9 the intersection of $\mathcal{P}$ and $\mathcal{R}$ is a line, say r . Likewise the intersection of $\mathcal{P}$ and $\mathcal{S}$ is a line, say s .



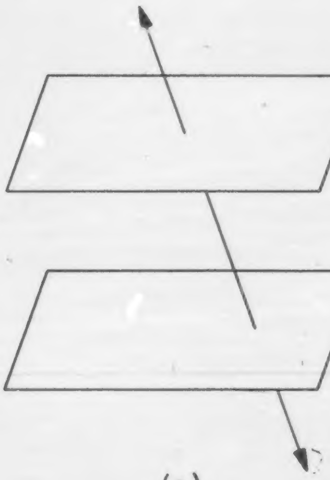
Moreover, these lines lie in the same plane, namely $\mathcal{P}$, and have no point in common, since the planes $\mathcal{R}$ and $\mathcal{S}$ have no point in common. Therefore, the intersections, r and s , are distinct parallel lines as asserted.

We should observe that the preceding theorem contains the hypothesis that the plane $\mathcal{P}$ intersects each of the parallel planes, $\mathcal{R}$ and $\mathcal{S}$. Actually, it is possible to prove the stronger result that if a plane intersects one of two distinct parallel planes (and does not coincide with it) then it intersects the second plane also and the intersections are parallel lines.

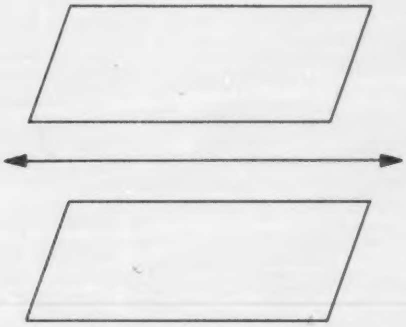
THEOREM 9-7. If a line intersects one of two distinct parallel planes in a single point, it intersects the other plane in a single point also.

THEOREM 9-8. If a line is parallel to one of two parallel planes, it is parallel to the other also.

The assertions of Theorems 9-7 and 9-8 are illustrated by Figures (a) and (b), respectively. We shall omit their proofs, however.



(a)



(b)

Problem Set 9-3

1. Make a sketch to illustrate the hypothesis of each of the following statements. Indicate whether each statement is True (T) or False (F).

   (a) If two distinct lines are parallel, every plane containing only one of them is parallel to the other line.

   (b) If two distinct lines are parallel, every line intersecting one of them intersects the other.

   (c) If two planes are parallel, any line in one of the planes is parallel to the other plane.

   (d) If two planes are parallel, any line in one of the planes is parallel to any line in the other plane.

   (e) If a plane and a line are both perpendicular to the same line, they are parallel to each other.

   (f) If a plane and a line are both parallel to the same line they are parallel to each other.

   (g) If each of two distinct parallel planes intersects a third plane, the lines of intersection are perpendicular.

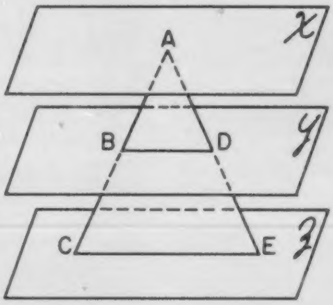
   (h) If two planes are parallel to the same line they are parallel to each other.

   (i) Two lines parallel to the same plane are parallel to each other.

   (j) If a plane intersects two intersecting planes, the lines of intersection may be parallel.

2. Hypothesis: Planes $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are parallel as shown, with $\overline{CE}$ in $\mathcal{Z}$, and A in $\mathcal{X}$. $\overline{AC}$ intersects $\mathcal{Y}$ at B and $\overline{AE}$ intersects $\mathcal{Y}$ at D. AC = CE.

   Prove: BD = BA.



3. Prove Theorem 9-5. (Hint: Let $\mathcal{P}$ be a plane which is parallel to one of two parallel lines, ℓ_1 and ℓ_2, say ℓ_1. Then one of the following must be true. (Why?)

   (a) $\mathcal{P}$ is parallel to ℓ_2.

   (b) $\mathcal{P}$ intersects ℓ_2 in a single point.

   Use Theorem 9-4 to prove that (b) is impossible.

9-4. Relations Involving Perpendicularity and Parallelism.

In Sections 9-2 and 9-3 we considered relations in space which involved, respectively, only perpendicularity and only parallelism. In this section we shall investigate configurations which involve both perpendicularity and parallelism. Since we shall omit the proofs of most of our theorems, you should perform carefully the following experiments and make sure that you understand and can visualize the relations they suggest.

Experiments

1. If two planes are parallel to a third plane, do you think they are parallel to each other? Illustrate by holding two books so that each is parallel to the top of your desk. Do the books appear to be parallel? Draw a diagram of three parallel planes.
2. Do you think there can be more than one plane which contains a given point and is parallel to a given plane? Why?
3. If two distinct planes are perpendicular to the same line, do you think they can intersect? Illustrate your conclusion by piercing two sheets of cardboard (or small sheets of paper) with a pencil. Draw a diagram of two distinct planes perpendicular to a given line.

4. Take a piece of cardboard, pierce it with your pencil and place it so that it appears perpendicular to the pencil at the midpoint of the pencil. Mark a point on the cardboard and find the distance from that point to each end of the pencil. Are the distances approximately the same? Choose another point and make a second measurement. Draw a diagram of a plane perpendicular to a segment at the midpoint of the segment.

5. If a line is parallel to a plane and is not contained in the plane, do you think all the perpendiculars joining the line to the plane are coplanar? Are these perpendicular segments equal in length? Are segments which are perpendicular to each of two distinct parallel planes and have their endpoints in the planes equal in length? Illustrate with a diagram.

THEOREM 9-9. Two planes which are perpendicular to the same line are parallel.

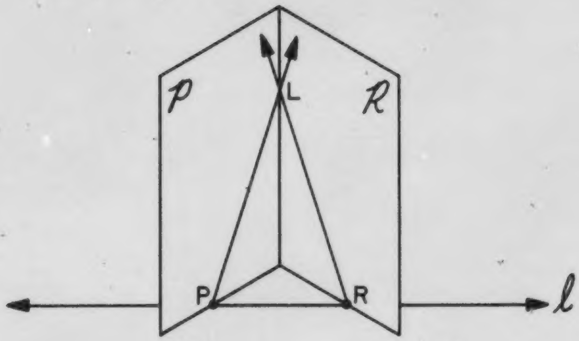
Proof: Let $\mathcal{P}$ and $\mathcal{R}$ be two planes each of which is perpendicular to a line ℓ . There are two possibilities to consider:

(a) $\mathcal{P}$ is parallel to $\mathcal{R}$.
(b) $\mathcal{P}$ intersects $\mathcal{R}$ in a line.

If we can prove the second case impossible, the theorem will be established.

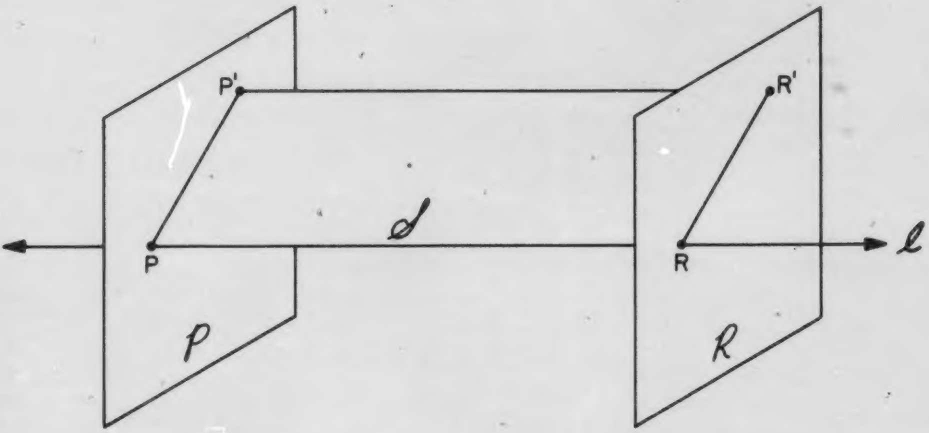
Suppose then, that $\mathcal{P}$ and $\mathcal{R}$ intersect in a line. Thus $\mathcal{P}$ and $\mathcal{R}$ are distinct. By Postulate 24 the points, say P and R , in which ℓ intersects the respective planes $\mathcal{P}$ and $\mathcal{R}$, must be distinct.

Let L be a point in both of the planes $\mathcal{P}$ and $\mathcal{R}$ but not on ℓ . From the definition of a plane perpendicular to a line, it follows that $\overleftrightarrow{LP}$ is perpendicular to ℓ at P and $\overleftrightarrow{LR}$ is perpendicular to ℓ at R . Hence, since P and R are distinct points, we have two lines each containing L and each perpendicular to ℓ .


$\mathcal{P}$
L
$\mathcal{R}$
P
R
ℓ

This is impossible, according to Theorem 5-11; hence the possibility that $\mathcal{P}$ and $\mathcal{R}$ intersect in a line leads to a contradiction and must be rejected. Thus $\mathcal{P}$ and $\mathcal{R}$ are parallel, as asserted.

THEOREM 9-10. If a line is perpendicular to one of two distinct parallel planes it is perpendicular to the other also.


P'
R'
$\mathcal{S}$
P
R
ℓ
$\mathcal{P}$
$\mathcal{R}$

Proof: Let $\mathcal{P}$ and $\mathcal{R}$ be two parallel planes, and let ℓ be a line which is perpendicular to one of them, say $\mathcal{P}$, at the point P . Then by Theorem 9-7, ℓ must also intersect $\mathcal{R}$ in a point, say R .

Let R' be any point of $\mathcal{R}$ distinct from R and let $\mathcal{S}$ be the plane determined by R' and ℓ . Then $\mathcal{S}$ intersects $\mathcal{R}$ in the line $\overleftrightarrow{RR'}$ and, by Theorem 9-6, must intersect $\mathcal{P}$ in a line, $\overleftrightarrow{PP'}$, which is parallel to $\overleftrightarrow{RR'}$.

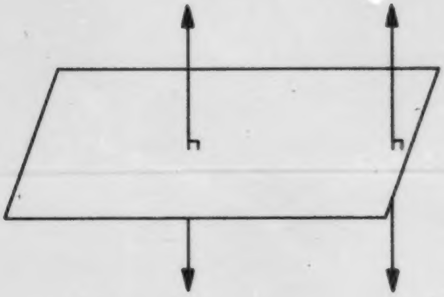
Thus in $\mathcal{S}$, ℓ is perpendicular to one of two parallel lines, namely $\overleftrightarrow{PP'}$ (Why?) and hence, it must be perpendicular to the other also. Since R' was any point in $\mathcal{R}$ distinct from R , it follows that ℓ is perpendicular to every line in $\mathcal{R}$ which contains R . Hence, by definition, ℓ is perpendicular to the plane $\mathcal{R}$, as asserted.

We shall omit the proofs of the remaining theorems in this section, but since some of them are asked for in the next problem set, it is necessary for us to introduce here the second of the postulates we referred to in Section 9-2. You will find that with this postulate, the missing proofs are not difficult to construct.

Postulate 25. Two lines which are perpendicular to the same plane are parallel.

THEOREM 9-11. If a plane is perpendicular to one of two distinct parallel lines, it is perpendicular to the other line also.

The assertion of this theorem is illustrated in the following figure:



THEOREM 9-12. If two lines are each parallel to a third line, they are parallel to each other.

This theorem completes the discussion we began in Chapter 6. There we showed that in a plane if each of two lines is parallel to a third line, they are parallel to each other. The present theorem assures that this result is true without the restriction that the three lines lie in the same plane.

THEOREM 9-13. Given a plane and a point not in the plane, there is a unique line which passes through the point and is perpendicular to the plane.

This theorem completes the discussion we began in Theorem 9-3. These two theorems together tell us that through any point there is a unique line which is perpendicular to a given plane.

The next two theorems describe properties of planes which are obvious counterparts of familiar properties of lines.

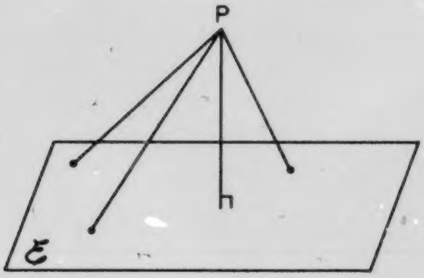
THEOREM 9-14. There is a unique plane parallel to a given plane through a given point.

THEOREM 9-15. If two planes are each parallel to a third plane, they are parallel to each other.

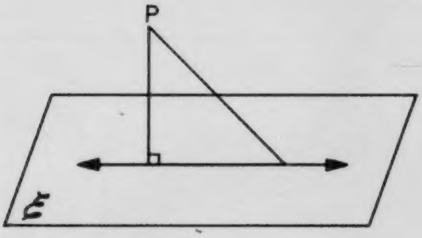
Theorems 9-12 and 9-15 provide the final steps in establishing that the relationship of parallelism has characteristic properties like those of equality, congruence, and similarity. The relationship of parallelism for lines in space has the reflexive, symmetric, and transitive properties. Likewise the relationship of parallelism for planes has the reflexive, symmetric, and transitive properties.

In Chapter 6 we considered a line and a point not on the line. We proved that the shortest segment joining the point to the line is the segment perpendicular to the line. As we might expect, a similar result holds in space.

Let $\mathcal{E}$ be a plane and P a point not on $\mathcal{E}$. There are many segments joining P to $\mathcal{E}$, in fact one for every point on $\mathcal{E}$. By Theorem 9-13, exactly one of these segments is perpendicular to $\mathcal{E}$.



THEOREM 9-16. The shortest segment joining a point to a plane not containing the point is the segment perpendicular to the given plane.



The proof of this theorem we shall leave as a problem. On the basis of this theorem, we formulate the following definition.

DEFINITION. The *distance between a point and a plane not containing the point* is the length of the segment joining the given point to the given plane and perpendicular to the given plane.

In Chapter 6 we proved that two parallel lines are everywhere equidistant, and the same property holds for parallel planes. More precisely, we have the following theorem.

THEOREM 9-17. All segments which are perpendicular to each of two distinct parallel planes and have their endpoints in the planes have the same length.

In Chapter 8 we showed that in a plane the set of all points which are equidistant from two given points P and Q is the line which is perpendicular to the segment $\overline{PQ}$ at its midpoint. The corresponding result in space geometry is the following:

THEOREM 9-18. The set of all points which are equidistant from the endpoints of a given segment is the plane which contains the midpoint of the segment and is perpendicular to the line which contains the segment.

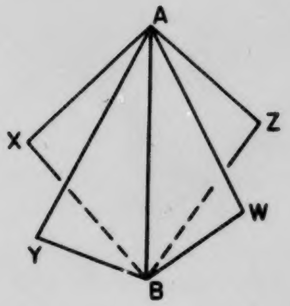
The proof of this theorem is deferred to Problem Set 9-7 where it will be an exercise in the use of coordinates in proof.

Problem Set 9-4

1. Assuming here that

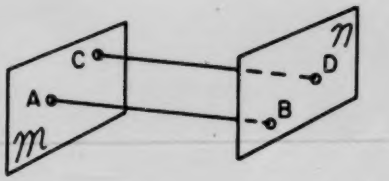
   $AX = BX$,
   $AY = BY$,
   $AW = BW$,
   $AZ = BZ$,

   why are W, X, Y, and Z coplanar?



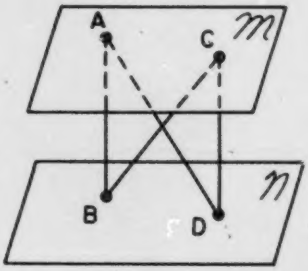
2. Hypothesis: A, C in $\mathcal{M}$; B, D in $\mathcal{N}$, $\mathcal{M} \perp \overline{AB}$, $\mathcal{N} \perp \overline{AB}$, $\mathcal{M} \perp \overline{CD}$.

   Prove: $\mathcal{N} \perp \overline{CD}$.



3. Hypothesis: In the figure $\mathcal{M} \parallel \mathcal{N}$, $\overline{AB} \perp \mathcal{N}$, $\overline{CD} \perp \mathcal{N}$.

   Prove: AD = CB .



4. Plane $\mathcal{E}$ is the perpendicular bisecting plane of $\overline{AB}$, as shown in the figure.

   (a) $\overline{AW} \cong$ _____ .

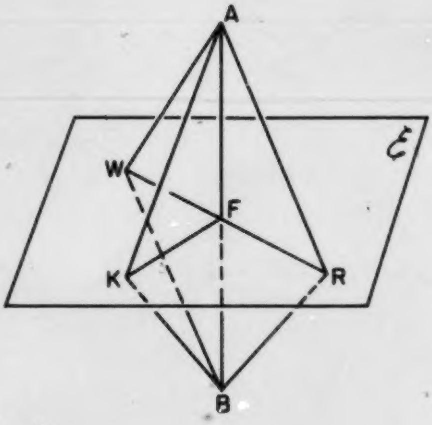
   $\overline{AK} \cong$ _____ .

   $\overline{AR} \cong$ _____ .

   m $\angle AFW =$ _____ .

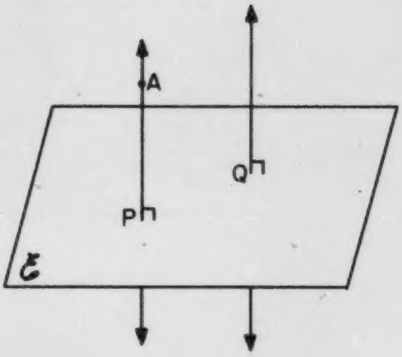
   $\angle AKF \cong$ _____ .

   (b) Does FW = FK = FR ? Explain.



Problems 5-8 are concerned with geometric projection. The following definitions are needed.

> DEFINITION. The projection of a point into a plane is the point of intersection of the given plane and the line which contains the given point and is perpendicular to the given plane.



Consider two examples in the diagram: P is the projection of A into $\mathcal{E}$; the point Q is in $\mathcal{E}$, and the projection of Q into $\mathcal{E}$ is Q itself.

DEFINITION. The *projection of a set of points into a plane* is the set of all points which are projections into the plane of points contained in the given set.

5. Using projection as defined above, answer the following.

   (a) Is the projection of a point always a point?

   (b) Is the projection of a segment always a segment?

   (c) Can the projection of an angle be a ray? a line? an angle?

   (d) Can the projection of an acute angle be an obtuse angle?

   (e) Is the projection of a right angle always a right angle?

   (f) Can the length of the projection of a segment be greater than the length of the segment?

   (g) If two segments are congruent will their projections be congruent?

   (h) If two lines do not intersect can their projections be two parallel lines?

   (i) If two lines do not intersect can their projections be two intersecting lines?

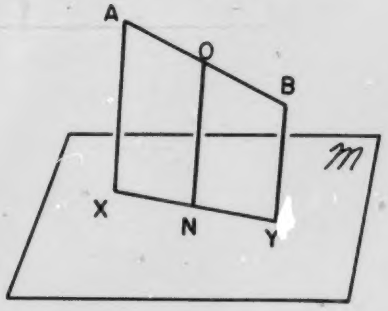
   (j) If two segments are parallel and congruent, will their projections be congruent?

6. Let the projection of point A into plane $\mathcal{M}$ be A', distinct from A. Let $\overrightarrow{AP}$ be the ray opposite to $\overrightarrow{AA'}$. Let B be a point such that the length of $\overline{AB}$ is 6 inches. Draw a diagram showing the projection of $\overline{AB}$ into $\mathcal{M}$, and find the length of the projection of $\overline{AB}$ into $\mathcal{M}$ in each of the following situations.

   (a) $\overline{AB} \parallel \mathcal{M}$.
   (b) $\overline{AB} \perp \mathcal{M}$.
   (c) $m\angle PAB = 30$.
   (d) $m\angle PAB = 45$.
   (e) $m\angle PAB = 60$.

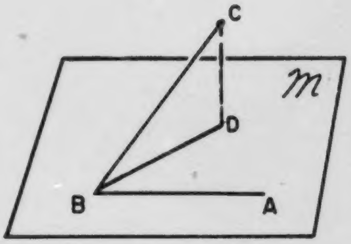
7. Given the figure with $\overline{AB}$ not in plane $\mathcal{M}$, $\overline{XY}$ the projection of $\overline{AB}$ into plane $\mathcal{M}$, O the midpoint of $\overline{AB}$, and N the projection of O into $\overline{XY}$.

   Prove that N is the midpoint of $\overline{XY}$.



8. Hypothesis: $\overline{BD}$ is the projection of $\overline{BC}$ into plane $\mathcal{M}$. $\overline{AB}$ lies in plane $\mathcal{M}$ and $\angle ABC$ is a right angle.

   Prove: $\angle ABD$ is a right angle. (Hint: Let $\overline{BE}$ be perpendicular to plane $\mathcal{M}$.)



9. Prove Theorem 9-16.

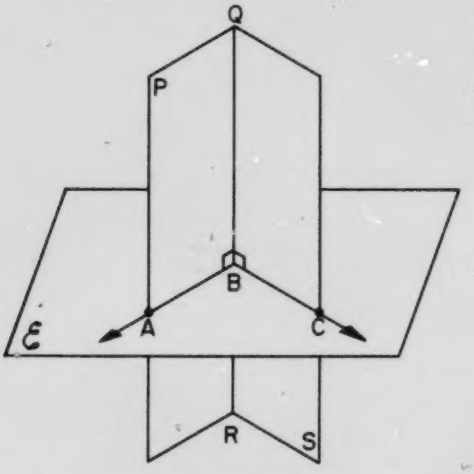
9-5. Dihedral Angles.

In Section 4-13 we introduced the notion of a dihedral angle via the following definition:

> DEFINITION. A dihedral angle is the union of a line and two halfplanes having this line as edge and not lying in the same plane.

At that time we were unable to assign measures to dihedral angles, but now that we have discussed perpendicularity and parallelism in space we can do so easily. First, however, we must reduce the problem to one involving plane angles, for which measures have already been defined.

The following figure shows a dihedral angle, namely $\angle P\text{-}QR\text{-}S$, and a plane, $\mathcal{E}$, which is perpendicular to the edge of the dihedral angle. We observe, in the diagram, that the intersection of the plane and the dihedral angle is the union of two rays, and furthermore that these two concurrent rays, namely $\overrightarrow{BA}$ and $\overrightarrow{BC}$, are not collinear. Thus the intersection is an angle.



DEFINITION. The intersection of a dihedral angle and any plane perpendicular to the edge of the given dihedral angle is called a plane angle of the dihedral angle.

If all plane angles of a dihedral angle were congruent, it would be natural to take their common measure as the measure of the dihedral angle itself. The next theorem guarantees that this can be done.

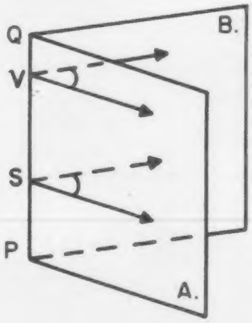
THEOREM 9-19. Any two plane angles of a dihedral angle are congruent.

Proof: Let S and V be the vertices of two distinct plane angles of the dihedral angle $\angle$A-PQ-B , (Figure (a)). Let U and W be points distinct from V on different sides of $\angle V$. In plane UVS , apply the Point-Plotting Theorem and locate point R on $\angle S$ such that

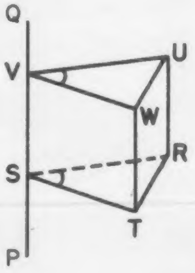
(1) $\overline{UV} \cong \overline{RS}$.

In plane WVS , locate point T on $\angle S$ such that

(2) $\overline{VW} \cong \overline{ST}$.



(a)



(b)

To prove the theorem, we must show that $\angle V \cong \angle S$. In order to do this, we shall apply the S.S.S. Congruence Postulate to show that $\triangle UVW \cong \triangle RST$.

Since $\angle V$ and $\angle S$ are plane angles of the dihedral angle, each of the planes UVW and RST is perpendicular to $\overleftrightarrow{PQ}$. By Theorem 9-9, the two planes are parallel to each other. By Theorem 9-6, $\overleftrightarrow{UV}$ and $\overleftrightarrow{RS}$ are parallel. This fact, together with (1), shows that UVSR is a parallelogram. Hence the segments $\overline{UR}$ and $\overline{VS}$ are both parallel and congruent.

A similar argument shows that WVST is a parallelogram, and therefore that the segments $\overline{VS}$ and $\overline{WT}$ are both parallel and congruent. By the transitive property of parallelism and congruence, the segments $\overline{UR}$ and $\overline{WT}$ are both parallel and congruent. In other words, UWTR is a parallelogram. Hence

(3) $\overline{UW} \cong \overline{RT}$.

Combining (1), (2), (3), the S.S.S. Congruence Postulate tells us that

$$\triangle UVW \cong \triangle RST .$$

Finally, $\angle V \cong \angle S$, and our proof is complete.

With the last theorem established, the measure of a dihedral angle can now be defined.

DEFINITION. The *measure of a dihedral angle* is the number which is the measure of any of its plane angles.

DEFINITION. A *right dihedral angle* is a dihedral angle whose measure is 90 .

DEFINITION. The planes determined by the faces of a right dihedral angle are said to be *perpendicular*.

The proofs of the following theorems about perpendicular planes are not difficult. Some have been left as problems.

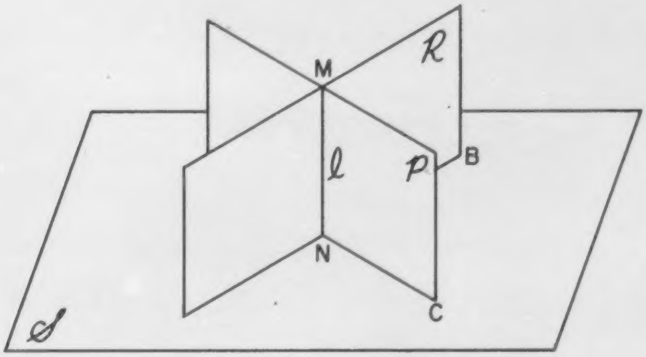
THEOREM 9-20. If a line is perpendicular to a plane, then any plane containing this line is perpendicular to the given plane.

THEOREM 9-21. If two planes are perpendicular, then any line in one of the planes which is perpendicular to their line of intersection is perpendicular to the other plane.

THEOREM 9-22. If two planes are perpendicular, then any line perpendicular to one of the planes at a point on their line of intersection lies in the other plane.

THEOREM 9-23. If two intersecting planes are each perpendicular to a third plane, then their line of intersection is perpendicular to this plane.

The assertion of the preceding theorem is illustrated in the following figure. Planes $\mathcal{P}$ and $\mathcal{R}$ are each perpendicular to the plane $\mathcal{S}$, and their line of intersection, ℓ, is therefore perpendicular to $\mathcal{S}$.



Problem Set 9-5

1. (a) How many dihedral angles are formed by the floor, walls, and ceiling of your classroom?
   (b) If two planes are perpendicular, what kind of dihedral angles are formed?
   (c) Give a definition of an
      (1) acute dihedral angle,
      (2) obtuse dihedral angle.

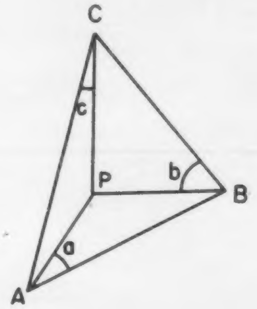
Draw three figures showing, respectively, an acute, a right, and an obtuse dihedral angle.

(d) Give a definition of adjacent dihedral angles. Illustrate with a drawing.

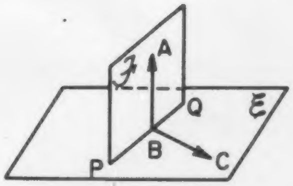
(e) Give a definition of supplementary dihedral angles. Illustrate with a drawing of a pair of adjacent supplementary angles.

(f) Give a definition of complementary dihedral angles. Illustrate with a drawing of a pair of adjacent complementary angles.

2. Each of $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ is perpendicular to the other two.
$m\angle a = m\angle b = m\angle c = 45$.
What is the measure of $\angle C\text{-}PA\text{-}B$? of $\angle CAB$?

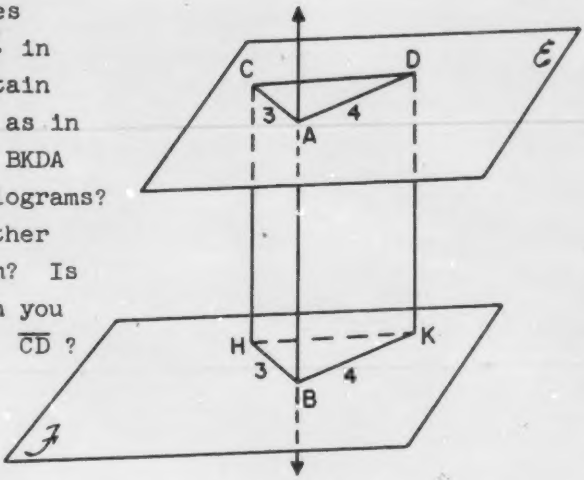


3. Prove Theorem 9-21.
Hypothesis: Referring to the figure on the right, $\mathcal{F} \perp \mathcal{E}$ and $\overleftrightarrow{AB} \perp \overleftrightarrow{PQ}$.
Prove $\overleftrightarrow{AB} \perp \mathcal{E}$.
(Hint: Take $\overleftrightarrow{BC} \perp \overleftrightarrow{PQ}$ in $\mathcal{E}$.)



4. Prove Theorem 9-23.
(Hint: Referring to the illustrative figure in the text, in plane $\mathcal{A}$ draw $\overleftrightarrow{XN} \perp \overleftrightarrow{NC}$ and $\overleftrightarrow{YN} \perp \overleftrightarrow{NB}$.)

5. Planes $\mathcal{E}$ and $\mathcal{F}$ are perpendicular to $\overleftrightarrow{AB}$. Lines $\overleftrightarrow{BK}$ and $\overleftrightarrow{BH}$, in plane $\mathcal{F}$, determine with $\overleftrightarrow{AB}$ two planes which intersect $\mathcal{E}$ in $\overleftrightarrow{AD}$ and $\overleftrightarrow{AC}$. Certain lengths are given, as in the figure. Are BKDA and BACH parallelograms? Can you give a further description of them? Is $\triangle BHK \cong \triangle ACD$? Can you give the length of $\overline{CD}$?



6. Prove: If a plane is perpendicular to the edge of a dihedral angle, then it is perpendicular to each face of the dihedral angle.

Review Problems

Chapter 9, Sections 1 to 5

1. In this problem, the symbol ℓ always denotes a line and the symbol $\mathcal{P}$ always denotes a plane. Fill in each blank with the one of the following words

always, sometimes, never

which makes the resulting statement true. In each case make a sketch to justify your answer.

(a) If ℓ_1 is parallel to ℓ_2 and ℓ_2 is parallel to ℓ_3, then ℓ_1 is ______ parallel to ℓ_3.

(b) If ℓ_1 is perpendicular to ℓ_2 and ℓ_2 is perpendicular to ℓ_3, then ℓ_1 is ______ perpendicular to ℓ_3.

(c) If $\mathcal{P}_1$ is perpendicular to $\mathcal{P}_2$ and $\mathcal{P}_2$ is perpendicular to $\mathcal{P}_3$, then $\mathcal{P}_1$ is _____ perpendicular to $\mathcal{P}_3$.

(d) If ℓ is perpendicular to $\mathcal{P}_1$ and $\mathcal{P}_1$ is parallel to $\mathcal{P}_2$, then ℓ is ______ perpendicular to $\mathcal{P}_2$.

(e) If ℓ_1 is parallel to $\mathcal{P}$ and $\mathcal{P}$ is parallel to ℓ_2 , then ℓ_1 is ______ parallel to ℓ_2 .

(f) If ℓ is parallel to $\mathcal{P}_1$ and $\mathcal{P}_1$ is perpendicular to $\mathcal{P}_2$, then ℓ is ______ parallel to $\mathcal{P}_2$.

(g) If $\mathcal{P}_1$ is perpendicular to ℓ and ℓ is perpendicular to $\mathcal{P}_2$, then $\mathcal{P}_1$ is ______ perpendicular to $\mathcal{P}_2$.

(h) If $\mathcal{P}_1$ is perpendicular to ℓ and ℓ is parallel to $\mathcal{P}_2$, then $\mathcal{P}_1$ is ______ perpendicular to $\mathcal{P}_2$.

(i) If $\mathcal{P}_1$ is perpendicular to $\mathcal{P}_2$ and $\mathcal{P}_2$ is perpendicular to $\mathcal{P}_3$, then $\mathcal{P}_1$ is ______ parallel to $\mathcal{P}_3$.

(j) If $\mathcal{P}$ is perpendicular to ℓ_1 and ℓ_1 is perpendicular to ℓ_2 , then $\mathcal{P}$ is ______ parallel to ℓ_2 .

2. Mark each of the following statements true (T) or false (F) .

(a) If a line is perpendicular to each of two distinct lines in a plane, it is perpendicular to the plane.

(b) If three distinct lines are perpendicular to the same line at the same point, the three lines are coplanar.

(c) Through a point not on a line, more than one plane can be passed perpendicular to the given line.

(d) Through a point not on a plane, only one line can be drawn parallel to the given plane.

(e) Through a point not on a plane, only one plane can be passed perpendicular to the given plane.

(f) If a plane is perpendicular to the edge of a dihedral angle, it is perpendicular to each face of the dihedral angle.

(g) If two planes are perpendicular, a line in one of the planes is perpendicular to the other plane.

(h) If two planes are perpendicular, a line perpendicular to one of the planes will lie in the other plane.

(i) If a plane intersects one of two distinct parallel lines, it intersects the other also.

(j) If two distinct lines are parallel, one and only one plane can be passed through one of these lines parallel to the other.

(k) If a line is parallel to one of two intersecting planes, it is parallel to their intersection.

(l) Two planes parallel to the same line are parallel.

(m) Through a line not perpendicular to a plane, a plane perpendicular to the given plane can be passed.

(n) The projection of a segment into a plane is a segment.

3. Which of the following lines or planes must be parallel? Which of them must coincide?

(a) Lines through the same point parallel to the same line.

(b) Lines perpendicular to the same plane.

(c) Lines perpendicular to the same line.

(d) Lines parallel to the same line.

(e) Lines parallel to the same plane.

(f) Planes perpendicular to the same line through the same point.

(g) Planes parallel to the same plane.

(h) Planes perpendicular to the same plane.

(i) Planes through the same point parallel to the same plane.

(j) Planes through the same point perpendicular to the same plane.

(k) Planes parallel to the same line.

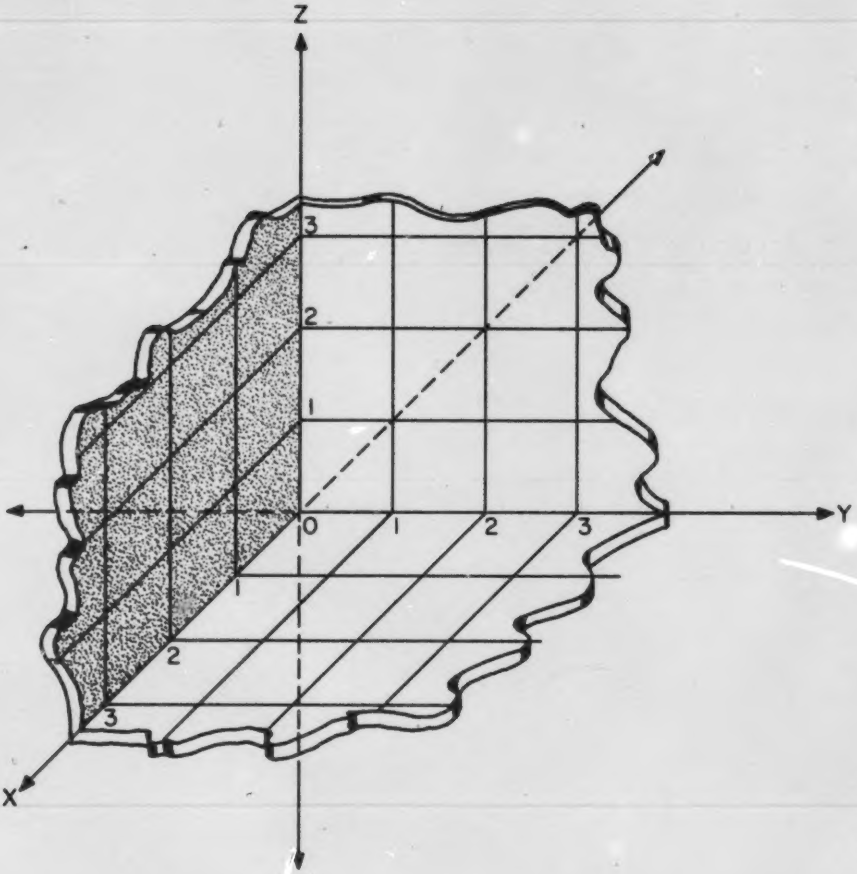
9-6. Coordinate Systems in Space.

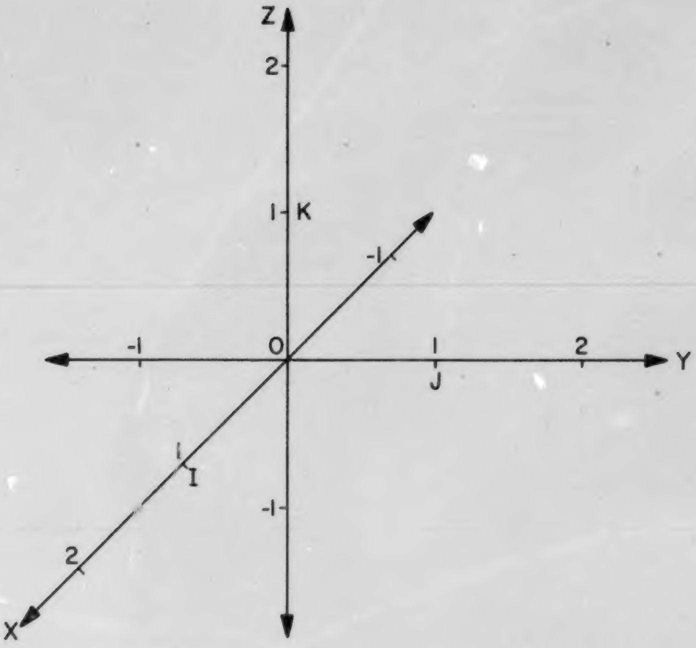
In Chapter 3 we introduced the fundamental idea of a coordinate system on a line, or a *one-dimensional coordinate system*, as it is sometimes called. In Chapter 8, we extended this idea to coordinate systems in a plane, or *two-dimensional coordinate systems*. Now that we have the necessary information about perpendicularity and parallelism in space, we are in a position to discuss coordinate systems in space, or *three-dimensional coordinate systems*. As we should expect, our development here will be very much like the development of two-dimensional coordinate systems in Chapter 8 and for this reason we shall omit many of the details and concentrate instead on the results themselves.

Let $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$ be any two perpendicular lines and let $\overleftrightarrow{OZ}$ be the unique line (Theorem 9-3) that is perpendicular to the plane of $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$ at their intersection, O . Clearly, then, each of the lines $\overleftrightarrow{OX}$, $\overleftrightarrow{OY}$, $\overleftrightarrow{OZ}$ is perpendicular to each of the other two, so that we have in fact three mutually perpendicular lines. Let I, J, and K be points on $\overleftrightarrow{OX}$, $\overleftrightarrow{OY}$ and $\overleftrightarrow{OZ}$ respectively such that

$$OI = OJ = OK = 1 .$$

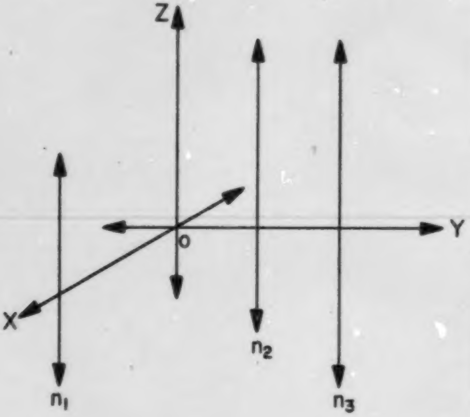
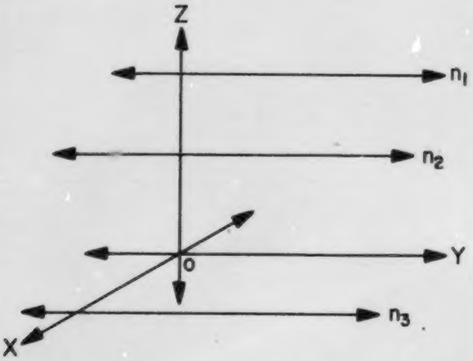
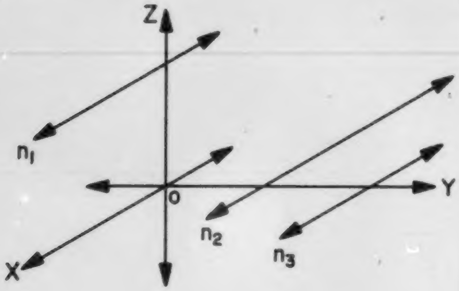
On the line $\overleftrightarrow{OX}$ there is a one-dimensional coordinate system with the point O as origin and the point I as unit point. On $\overleftrightarrow{OY}$, there is a one-dimensional coordinate system with the point O as origin and the point J as unit point. On $\overleftrightarrow{OZ}$ there is a one-dimensional coordinate system with the point O as origin and the point K as unit point. We shall refer to these coordinate systems as the x- , y- , and z-coordinate systems, respectively.


Z
Y
X

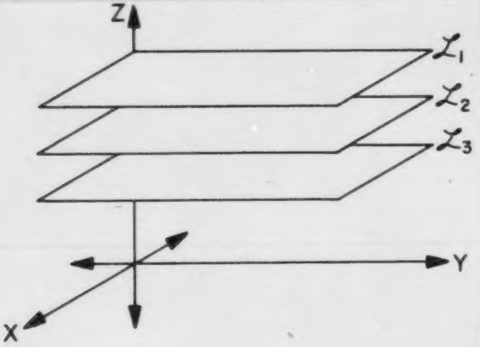
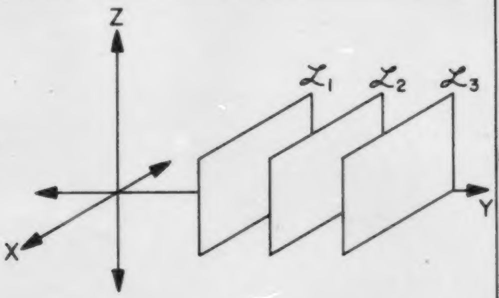
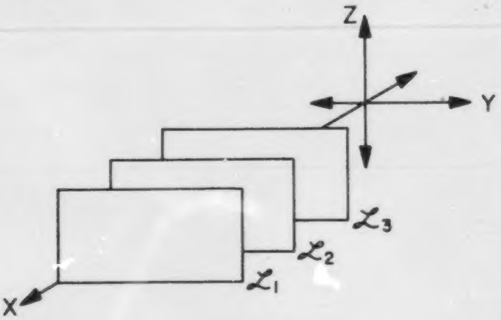


The line $\overleftrightarrow{OX}$ is called the x-axis, the line $\overleftrightarrow{OY}$ is called the y-axis, the line $\overleftrightarrow{OZ}$ is called the z-axis. Collectively $\overleftrightarrow{OX}$, $\overleftrightarrow{OY}$, $\overleftrightarrow{OZ}$ are called the coordinate axes. The point 0 , which is common to the three coordinate axes, is called the origin. The plane containing the x-axis and the y-axis is called the xy-plane, the plane containing the x-axis and the z-axis is called the xz-plane, the plane containing the y-axis and the z-axis is called the yz-plane. Collectively these planes are called the coordinate planes.

From the theorems we proved in Section 9-4, it is clear that all lines parallel to the z-axis are perpendicular to the xy-plane. Similarly, all lines parallel to the y-axis are perpendicular to the xz-plane, and all lines parallel to the x-axis are perpendicular to the yz-plane. Using the convenient set-builder notation, these important observations can be summarized as follows:

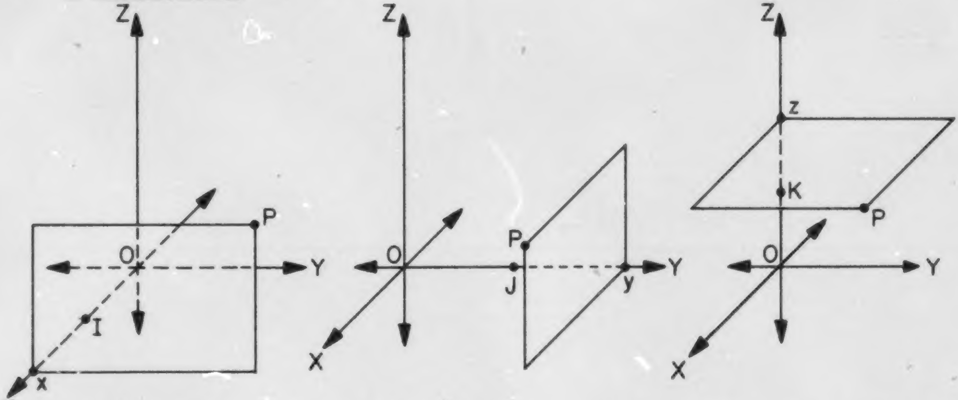
"Pictorial" description	"Parallel" description	"Perpendicular" description
 Z O Y X n_1 n_2 n_3	$\{n : n \parallel \text{z-axis}\}$	$\{n : n \perp \text{xy-plane}\}$
 Z n_1 n_2 O Y n_3 X	$\{n : n \parallel \text{y-axis}\}$	$\{n : n \perp \text{xz-plane}\}$
 Z n_1 O Y n_2 n_3 X	$\{n : n \parallel \text{x-axis}\}$	$\{n : n \perp \text{yz-plane}\}$

It should be clear also that (a) all planes parallel to the xy-plane are perpendicular to the z-axis, (b) all planes parallel to the xz-plane are perpendicular to the y-axis, (c) all planes parallel to the yz-plane are perpendicular to the x-axis.

"Pictorial" description	"Parallel" description	"Perpendicular" description
 Z $\mathcal{L}_1$ $\mathcal{L}_2$ $\mathcal{L}_3$ Y X	$\{\mathcal{L}:\mathcal{L} \parallel \text{xy-plane}\}$	$\{\mathcal{L}:\mathcal{L} \perp \text{z-axis}\}$
 Z $\mathcal{L}_1$ $\mathcal{L}_2$ $\mathcal{L}_3$ Y X	$\{\mathcal{L}:\mathcal{L} \parallel \text{xz-plane}\}$	$\{\mathcal{L}:\mathcal{L} \perp \text{y-axis}\}$
 Z Y $\mathcal{L}_3$ $\mathcal{L}_2$ $\mathcal{L}_1$ X	$\{\mathcal{L}:\mathcal{L} \parallel \text{yz-plane}\}$	$\{\mathcal{L}:\mathcal{L} \perp \text{x-axis}\}$

As you will remember from Chapter 3, a coordinate system on a line is a one-to-one correspondence between the line and the set of real numbers. Similarly, as you learned in Chapter 8, a coordinate system in a plane is a one-to-one correspondence between the plane and the set of ordered pairs of real numbers. Now we are going to establish a coordinate system in space as a one-to-one correspondence between space and the set of ordered triples of real numbers.

To do this, let P be any point in space. Then through P there passes a unique plane which is perpendicular to the x-axis (Postulate 24). This plane intersects the x-axis in a point which has a coordinate, say x , in the one-dimensional coordinate system established on $\overleftrightarrow{OX}$ by the ordered pair (O,I) . This number, x , we define to be the first coordinate, or x-coordinate, of P .



Similarly, through P there passes a unique plane which is perpendicular to the y-axis and this plane intersects the y-axis in a point which has a coordinate, say y , in the coordinate system determined on $\overleftrightarrow{OY}$ by the ordered pair (O,J) . The number y we define to be the second coordinate, or y-coordinate, of P . Finally, through P there passes a unique plane perpendicular to the z-axis, and this plane intersects the z-axis in a point which has a coordinate, say z , in the coordinate system determined on $\overleftrightarrow{OZ}$ by the ordered pair (O,K) . The number z we define to be the third coordinate, or z-coordinate, of P .

Conversely, if any ordered triple, say (x',y',z') is given, there is a unique point P' having (x',y',z') as its coordinates. In fact, there is a unique plane perpendicular to the x-axis at the point whose x-coordinate is x', and there is a unique plane perpendicular to the y-axis at the point whose y-coordinate is y'. These two planes cannot be parallel (Why?), hence they must intersect in a line, m, which by Theorem 9-23, is perpendicular to the xy-plane. Finally, there is a unique plane which is perpendicular to the z-axis at the point whose z-coordinate is z'. Since m is perpendicular to this plane (Why?), it must intersect it in a point P', whose coordinates are clearly (x',y',z'), as required.

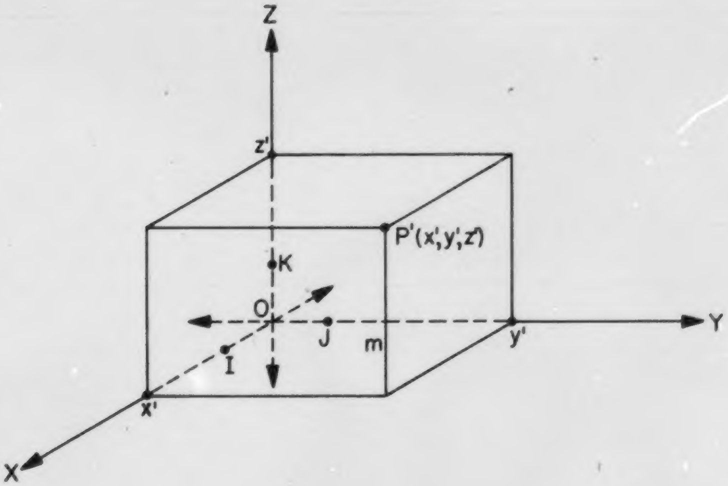


Figure (a)

As our efforts in this chapter have already illustrated, it is difficult to represent space configurations by drawings on a sheet of paper. You ought to practice plotting in a three-dimensional coordinate system so that you can make drawings and visualize the space relations which they suggest. Of course, you should begin your practice with simple situations, such as plotting a single point, a pair of points, a segment, or a line.

Figure (a) shows how a single point may be plotted. Figures (b) and (c) below show a segment and a line. Notice that an essential technique in plotting is the ability to draw a line parallel to a coördinate axis. Notice also the significance of perspective along the x-axis. You may wish to refer again to Appendix V for further help.

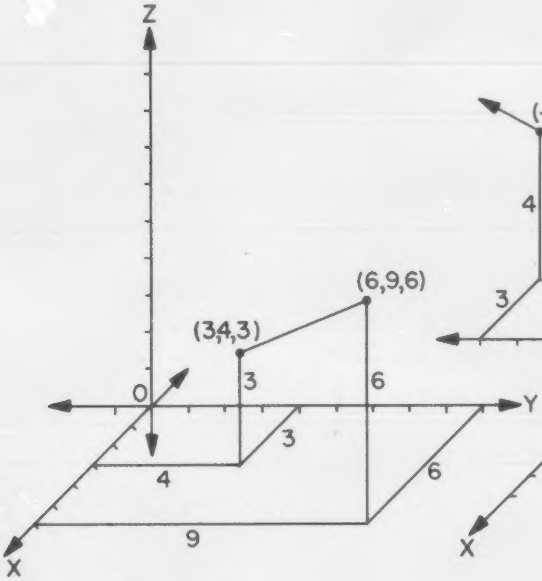


Figure (b)

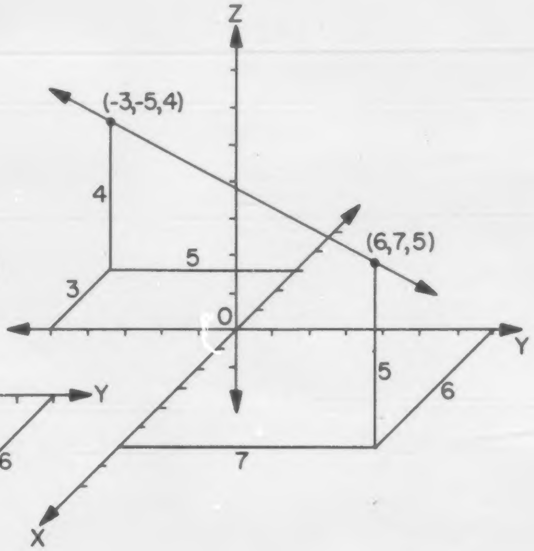


Figure (c)

By Postulate 24, it is clear that the procedure which determines the coordinates of a point, P , works just as well when P is in one or more of the coordinate planes as it does when P does not lie in any of the coordinate planes. Hence, it should be easy for you to verify the results which are summarized in the following table.

Set of points	Form of the coordinates of any point of the set	Equation(s) satisfied by the coordinates of any point of the set
origin	(0,0,0)	$x = 0$ and $y = 0$ and $z = 0$
x-axis	(x,0,0)	$y = 0$ and $z = 0$
y-axis	(0,y,0)	$x = 0$ and $z = 0$
z-axis	(0,0,z)	$x = 0$ and $y = 0$
xy-plane	(x,y,0)	$z = 0$
xz-plane	(x,0,z)	$y = 0$
yz-plane	(0,y,z)	$x = 0$

From the definition of the x-coordinate of a point, P, it is clear that all points which lie in a particular plane, $\mathcal{P}$, perpendicular to the x-axis have the same x-coordinate, say $x = x_1$. It is also clear that, conversely, any point whose x-coordinate is x_1 must lie in the plane $\mathcal{P}$. In other words, the coordinates of every point in a plane which is perpendicular to the x-axis satisfy an equation of the form $x = x_1$, and conversely, any point whose coordinates satisfy this equation lies in this plane.

Similarly, we can say that all points which lie in a plane perpendicular to the y-axis have the same y-coordinate, say $y = y_1$, or in other words have coordinates which satisfy the equation $y = y_1$, and conversely.

What do these observations tell us about the coordinates of points which lie on the line of intersection of a plane perpendicular to the x-axis and a plane perpendicular to the y-axis? Do you see that if a point P lies simultaneously in a plane whose points have coordinates satisfying the equation $x = x_1$ and in a plane whose points have coordinates satisfying the equation $y = y_1$, then the coordinates of P must satisfy both of these conditions? If you understand this, it should not be hard for you to verify the assertions in the following table.

Set of points	Form of the coordinates of any point of the set	Equation(s) satisfied by the coordinates of any point of the set
plane $\perp$ x-axis	(x_1,y,z)	$x = x_1$
plane $\perp$ y-axis	(x,y_1,z)	$y = y_1$
plane $\perp$ z-axis	(x,y,z_1)	$z = z_1$
line $\perp$ xy-plane	(x_1,y_1,z)	$x = x_1$ and $y = y_1$
line $\perp$ xz-plane	(x_1,y,z_1)	$x = x_1$ and $z = z_1$
line $\perp$ yz-plane	(x,y_1,z_1)	$y = y_1$ and $z = z_1$

Problem Set 9-6

1. Using the same set of axes, plot the points P, Q, R, S, T, U, V : $P(0,1,0)$; $Q(-3,0,0)$; $R(-3,1,0)$; $S(-3,1,4)$; $T(3,1,4)$; $U(3,-1,4)$; $V(3,-1,-4)$.

2. Using the same set of axes, plot the points A, B, C, D, E : $A(0,-1,3)$; $B(3,4,6)$; $C(-4,2,-7)$; $D(1,-3,0)$; $E(5,2,-4)$.

3. Describe the location of all the points in space for which

   (a) $x = 0$.
   (b) $x = 2$.
   (c) $x = -3$.

   Illustrate with a sketch for each part.

4. Sketch the set of all points in space which satisfy the given condition.

   (a) $y = 0$. (c) $z = 0$.
   (b) $y = 2$. (d) $z = -4$.

5. Describe the set of points represented by each of the following:

   (a) $\{(x,y,z):y = z = 0\}$.
   (b) $\{(x,y,z):x = y = 0\}$.
   (c) $\{(x,y,z):x = z = 0\}$.
   (d) $\{(x,y,z):x = y = z = 0\}$.
   (e) $\{(x,y,z):y = 2 \text{ and } x = 0\}$.
   (f) $\{(x,y,z):x = 2 , y = 1\}$.

6. Suggest a convenient set of coordinates for the eight vertices:
   (a) of a cube each of whose edges has length a ;
   (b) of a rectangular solid (parallelepiped) having mutually perpendicular edges of lengths a, b, c , respectively.
7. Where are all the points in space for which $x + y = 2$? Sketch the graph.

9-7. The Distance Formula in Space.

In Section 8-2, we proved two theorems (Theorems 8-1 and 8-2) which enable us to determine the distance between any two points on a line parallel to either of the coordinate axes. Similar results hold in space, and we have, specifically:

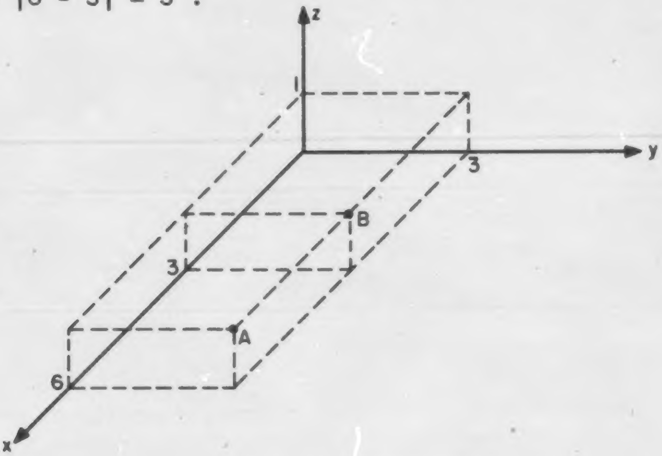
THEOREM 9-24. If P_1 and P_2 are points on a line parallel to the x-axis, then $P_1P_2 = |x_1 - x_2|$, where x_1 and x_2 are the x-coordinates of P_1 and P_2 , respectively.

THEOREM 9-25. If P_1 and P_2 are points on a line parallel to the y-axis, then $P_1P_2 = |y_1 - y_2|$, where y_1 and y_2 are the y-coordinates of P_1 and P_2 , respectively.

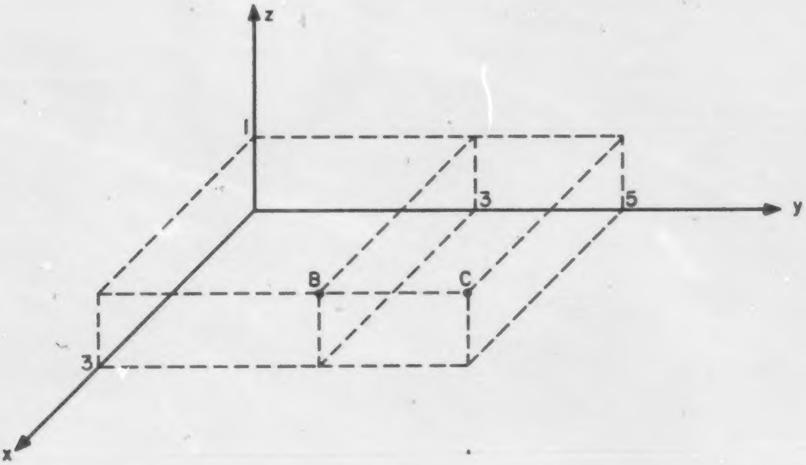
THEOREM 9-26. If P_1 and P_2 are points on a line parallel to the z-axis, then $P_1P_2 = |z_1 - z_2|$, where z_1 and z_2 are the z-coordinates of P_1 and P_2, respectively.

The proofs of these theorems are very easy and we shall omit them. Two applications of these theorems are illustrated in the following figures.

(a) If $A = (6,3,1)$ and $B = (3,3,1)$, then $AB = |6 - 3| = 3$.



(b) If $C = (3,5,1)$ and $B = (3,3,1)$, then $BC = |3 - 5| = 2$.

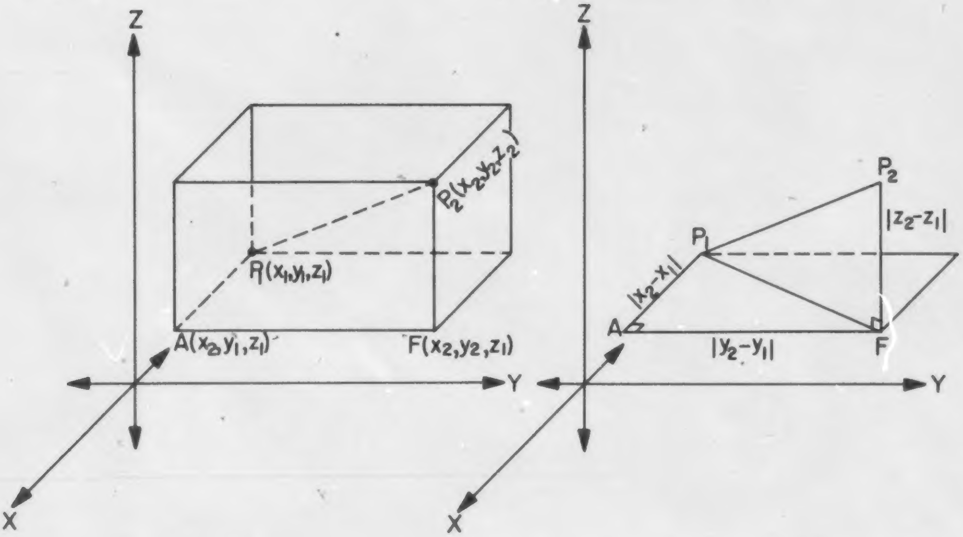


As one of the important applications of plane coordinate systems, we developed the so-called distance formula

$$P_2P_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

which enables us to find the distance between any two points, $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, in a plane. It is now natural to seek a formula, analogous to the distance formula in a plane, which will express the distance between any two points in space in terms of the coordinates of the points.

To do this, let $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ be any two points in space, and consider the figure which is formed by the three planes through P_1 which are respectively perpendicular to the coordinate axes and the three planes through P_2 which are respectively perpendicular to the coordinate axes. Let A and F be the points (x_2,y_1,z_1) and (x_2,y_2,z_1) .



5

First, suppose that P_1 and P_2 do not lie in a plane which is perpendicular to one of the coordinate axes. Clearly, $\overleftrightarrow{P_1A}$ is perpendicular to the yz-plane (Theorem 9-23) and hence parallel to the x-axis (Postulate 25). Similarly $\overleftrightarrow{AF}$ is perpendicular to the xz-plane and hence parallel to the y-axis, and $\overleftrightarrow{FP_2}$ is perpendicular to the xy-plane and hence parallel to the z-axis. Therefore $\angle P_1AF$ and $\angle P_1FP_2$ are right angles (why?). Hence, applying the Theorem of Pythagoras to the two right triangles $\triangle P_1FP_2$ and $\triangle P_1AF$, we obtain respectively

$$(P_2P_1)^2 = (FP_1)^2 + (P_2F)^2$$

and

$$(FP_1)^2 = (AP_1)^2 + (FA)^2 .$$

Now substituting for $(FP_1)^2$ from the second of these equations into the first, we get

$$(P_2P_1)^2 = [(AP_1)^2 + (FA)^2] + (P_2F)^2$$

$$P_2P_1 = \sqrt{(AP_1)^2 + (FA)^2 + (P_2F)^2} .$$

But according to Theorems 9-24, 9-25, and 9-26, respectively,

$$AP_1 = |x_2 - x_1| ,$$

$$FA = |y_2 - y_1| ,$$

$$P_2F = |z_2 - z_1| .$$

Therefore, substituting,

$$P_2P_1 = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2}$$

or, since $|q|^2 = q^2$ for every real number q,

$$P_2P_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} .$$

If P_1 and P_2 lie in a plane which is perpendicular to one of the coordinate axes, then either $x_2 = x_1$, $y_2 = y_1$, or $z_2 = z_1$, and one or more of the terms under the radical in the last formula is zero. It is easy to see that the formula is still valid. For instance, if $z_2 = z_1$, then $P_2 = F$, and clearly the correct expression for P_2P_1 is

$$P_2P_1 = \sqrt{(AP_1)^2 + (FA)^2}$$
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$

Likewise, if P_1 and P_2 determine a line which is perpendicular to one of the coordinate planes, then two of the coordinates of P_1 must equal the corresponding coordinates of P_2 , and only one of the terms under the radical in the formula for P_2P_1 is different from zero. Again the formula is valid. For instance, if $y_2 = y_1$ and $z_2 = z_1$, then $P_2 = A$ and the correct expression for P_2P_1 is

$$P_2P_1 = AP_1 = |x_2 - x_1| = \sqrt{(x_2 - x_1)^2} .$$

Finally, if $P_2 = P_1$, then $x_2 = x_1$, $y_2 = y_1$, $z_2 = z_1$ and every term in the formula for P_2P_1 is zero. Thus $P_2P_1 = 0$, as of course it should if $P_2 = P_1$. Hence, in every case, the three-dimensional distance formula

$$P_2P_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

gives us the distance between the points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$. We state this result as a theorem.

THEOREM 9-27. The distance between the points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ is given by

$$P_2P_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} .$$

Example 1

What is the distance between the points $P_1(2,4,0)$ and $P_2(1,2,-2)$?

By direct substitution into the three-dimensional distance formula, we obtain

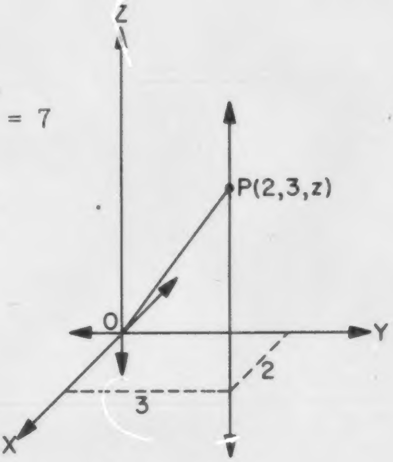
$$\begin{aligned} P_2P_1 &= \sqrt{(1-2)^2 + (2-4)^2 + (-2-0)^2} \\ &= \sqrt{1+4+4} \\ &= 3 . \end{aligned}$$

Example 2

Find the points which lie on the line perpendicular to the xy-plane at the point $(2,3)$ and are at a distance of 7 from the origin.

Clearly, any point which lies on the line perpendicular to the xy-plane at the point $(2,3)$ has x-coordinate 2 and y-coordinate 3 . Hence, a point on this line is determined as soon as its z-coordinate is known. Thus, we must determine the value, or values, of z such that the distance from the origin, $O(0,0,0)$, to the point $P(2,3,z)$ is 7 . Using the distance formula we thus have

$$\begin{aligned} PO = \sqrt{(2-0)^2 + (3-0)^2 + (z-0)^2} &= 7 \\ \sqrt{4+9+z^2} &= 7 \\ \sqrt{z^2+13} &= 7 \\ z^2 + 13 &= 49 \\ z^2 &= 36 \\ z &= \pm 6 \end{aligned}$$



Thus there are two points which meet the requirements of the problem, namely $P_1(2,3,6)$ and $P_2(2,3,-6)$, as can be checked immediately.

Problem Set 9-7

1. Find the distance between the points P_1 and P_2 if the coordinates of P_1 and P_2 are as follows:
   - (a) $(4,-1,-5)$; $(7,3,7)$.
   - (b) $(0,4,5)$; $(-6,2,8)$.
   - (c) $(3,0,7)$; $(-1,3,7)$.
   - (d) $(3,4,5)$; $(8,4,1)$.
   - (e) $(0,1,0)$; $(-1,-1,-2)$.
   - (f) $(1,2,3)$; $(0,0,0)$.
2. A line m is perpendicular to the yz-plane at the point $P(0,3,4)$. Find the points which lie on line m and are at a distance of 13 from the origin.
3. A line q is perpendicular to the xy-plane at the point $P(6,8,0)$. Find the points which lie on line q and are at a distance of 10 from the origin.
4. A line ℓ is perpendicular to the xz-plane and contains the point $P(1,-2,1)$. Find the points of ℓ which are at a distance of 4 from the origin.
5. Show that $\triangle ABC$ with vertices $A(2,4,1)$, $B(1,2,-2)$, $C(5,0,-2)$ is a right triangle.
6. Is the triangle with vertices $A(2,0,8)$, $B(8,-4,6)$, $C(-4,-2,4)$ isosceles? Justify your answer.
7. Given the vertices of two triangles, $\triangle ABC$ and $\triangle DEF$; for each of the triangles, determine if it is equilateral.

   $A(1,3,3)$, $B(2,2,1)$, $C(3,4,2)$;
   $D(6,2,3)$, $E(1,-3,2)$, $F(0,-2,-5)$.
8. (a) Show that the opposite sides of the figure ABCD with vertices $A(3,2,5)$, $B(1,1,1)$, $C(4,0,3)$, $D(6,1,7)$ are congruent.

   (b) Does this prove that the figure is a parallelogram? Explain.

9. (a) Show that the opposite sides of the figure ABCD with vertices $A(5,1,1)$, $B(3,1,0)$, $C(4,3,-2)$, $D(6,3,-1)$ are congruent.
   (b) Show that the angles of the figure in Part (a) are all right angles.
   (c) Do the results in (a) and (b) prove that the figure is a rectangle? Explain.

10. Using coordinates, prove Theorem 9-18. (Hint: This follows closely the proof of the corresponding theorem in a plane.)

9-8. Parametric Equations of a Line in Space.

In Section 8-7, we obtained what we called parametric equations of the line determined by two distinct points, $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, namely

$$x = x_1 + k(x_2 - x_1)$$

$$y = y_1 + k(y_2 - y_1) .$$

For every value of k, the corresponding numbers, x and y, are the coordinates of a point on $\overleftrightarrow{P_1P_2}$ and, conversely, to every point on $\overleftrightarrow{P_1P_2}$ there corresponds a unique value of the parameter k such that these equations give the coordinates of the point. By an argument similar to the one which establishes the result in the plane, it is possible to establish the corresponding result for space.

THEOREM 9-28. If $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ are any two distinct points, then for every value of k the point whose coordinates are

$$x = x_1 + k(x_2 - x_1)$$

$$y = y_1 + k(y_2 - y_1)$$

$$z = z_1 + k(z_2 - z_1)$$

lies on $\overleftrightarrow{P_1P_2}$ and, conversely, to every point on $\overleftrightarrow{P_1P_2}$ there corresponds a unique value of k such that these equations give the coordinates of the point.

Example 1

What are the coordinates of the point in which the line determined by $P_1(3,7,2)$ and $P_2(1,1,-2)$ intersects the xy-plane?

We know that a point is in the xy-plane if and only if its z-coordinate is zero. Our problem is to find the point on $\overleftrightarrow{P_1P_2}$ whose z-coordinate is zero. Now, by the last theorem, the coordinates of any point on $\overleftrightarrow{P_1P_2}$ are given by the equations

$$x = 3 + k(1 - 3) = 3 - 2k \,,$$
$$y = 7 + k(1 - 7) = 7 - 6k \,,$$
$$z = 2 + k(-2 - 2) = 2 - 4k \,.$$

Hence, for the point whose z-coordinate is zero, we must have

$$2 - 4k = 0 \quad \text{or} \quad k = \frac{1}{2} \,.$$

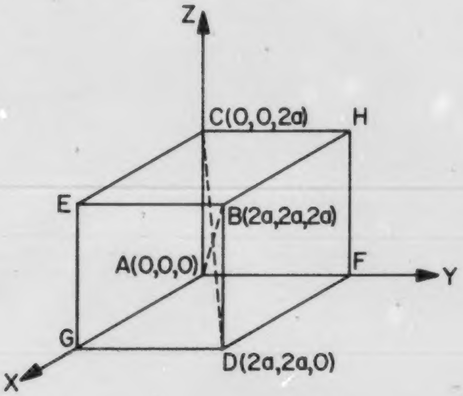
Substituting this value into the formulas for x and y, we find

$$x = 3 - 2(\tfrac{1}{2}) = 2 \,,$$
$$y = 7 - 6(\tfrac{1}{2}) = 4 \,.$$

The required point is the point with coordinates $(2,4,0)$.

Example 2

Show that the diagonals of a cube (a) have equal lengths, (b) bisect each other and (c) are not perpendicular to each other.



Proof: Let the length of each edge of the cube be $2a$. Choose the coordinate axes so that one vertex, A, is at the origin and three edges lie in the positive x-, y-, z-axis. Then, the endpoints of diagonal $\overline{AB}$ are $A(0,0,0)$ and $B(2a,2a,2a)$. Another diagonal, $\overline{CD}$, has endpoints $C(0,0,2a)$ and $D(2a,2a,0)$.

(a) $AB = \sqrt{(2a - 0)^2 + (2a - 0)^2 + (2a - 0)^2} = \sqrt{12a^2} = 2a\sqrt{3}$,

$CD = \sqrt{(2a - 0)^2 + (2a - 0)^2 + (0 - 2a)^2} = \sqrt{12a^2} = 2a\sqrt{3}$.

Therefore, $AB = CD$. Similarly, $AB = GH = EF$.

The length of each diagonal of a cube is $\sqrt{3}$ times the length of each edge.

(b) $\overleftrightarrow{AB} = \{(x,y,z): x = 0 + 2ak,\ y = 0 + 2ak,\ z = 0 + 2ak,$ k is a real number$\}$.

By taking $k = \frac{1}{2}$, we find the midpoint of $\overline{AB}$ to be

(a,a,a).

Similarly, $\overleftrightarrow{CD} = \{(x,y,z): x = 0 + 2ah,\ y = 0 + 2ah,$ $z = 2a - 2ah,\ h$ is a real number$\}$.

By taking $h = \frac{1}{2}$, we find the midpoint of $\overline{CD}$ to be

(a,a,a).

Similarly, we find that the midpoint of $\overline{EF}$ is (a,a,a) and the midpoint of $\overline{GH}$ is (a,a,a).

Thus, any two of the diagonals bisect each other.

(c) If M is the common midpoint of the diagonals, then, by Part (a), $AM = \frac{1}{2} \cdot AB = a\sqrt{3}$ and $CM = a\sqrt{3}$. Thus $(AM)^2 = 3a^2$ and $(CM)^2 = 3a^2$, but $(AC)^2 = (2a)^2 = 4a^2$. Therefore, $(AM)^2 + (MC)^2 \neq (AC)^2$. Hence, $\overleftrightarrow{AM}$ and $\overleftrightarrow{CM}$ are not perpendicular.

By this same reasoning all other pairs of diagonals can be proved not perpendicular.

Problem Set 9-8

1. In Example 2 above, prove $GH = EF = AB$.

2. Given points $A(-2,0,4)$, and $B(8,2,-2)$, use set notations and parametric equations to express

   (a) $\overleftrightarrow{AB}$. (b) $\overline{AB}$. (c) $\overrightarrow{AB}$.

3. (a) Find the midpoint of $\overline{AB}$ in Problem 2.
   (b) Find the trisection point of $\overline{AB}$ nearer A.
   (c) Find the trisection point of $\overline{AB}$ nearer B.
   (d) Find P if P is in $\overrightarrow{AB}$ and $AP = 3AB$.
   (e) Find P if P is in the ray opposite to $\overrightarrow{AB}$ and $AP = 3AB$.
   (f) Find the coordinates of the point in which $\overleftrightarrow{AB}$ intersects the xy-plane; the xz-plane; the yz-plane.
   (g) Find the coordinates of the point in which $\overleftrightarrow{AB}$ intersects the plane whose equation is $z = 3$; whose equation is $y = -2$; whose equation is $x = -3$.

4. Prove that the diagonals of a rectangular solid are equal in length and that they bisect each other.

5. Show that $A(-1,5,3)$, $B(1,4,4)$ and $C(5,2,6)$ are collinear.

6. What are the coordinates of the point P in which the line determined by $P_1(2,1,3)$ and $P_2(3,-2,1)$ intersects the yz-plane?

7. What are the coordinates of the point P in which the line determined by $P_1(-1,2,-1)$ and $P_2(3,-2,2)$ intersects the xz-plane?

8. A rectangular solid has three adjacent faces in the coordinate planes. One vertex is at the origin and another has coordinates (2a,2b,2c) . What are the possible relationships among a, b, c if two of the diagonals are perpendicular to each other?

9. (a) Given the points A(7,1,3) , C(4,-2,3) , find the coordinates of the midpoint, M , of $\overline{AC}$.
   (b) Consider the points B(5,0,0) and D(6,y,z) . Find y and z so that the midpoint of $\overline{BD}$ is the same point M as in Part (a).
   (c) Is figure ABCD a parallelogram? Explain.

10. Using ideas of midpoints, as in Problem 9 above, show that the figure in Problem 8 of Problem Set 9-7 _is_ a parallelogram.

11. Using ideas of midpoints, as in Problem 9 above, show that the figure in Problem 9 of Problem Set 9-7 _is_ a rectangle.

9-9. Equation of a Plane.

Our study of the equations of lines in both two-and three-dimensional coordinate systems raises the question of whether or not planes in space can likewise be characterized by equations. The answer is Yes , and we shall conclude the chapter by finding an equation corresponding to a plane. First, however, it is convenient to introduce the following definition.

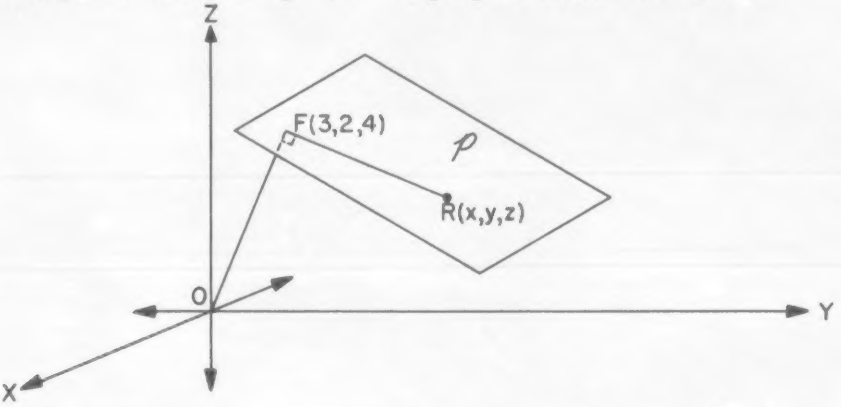
DEFINITION. If $\mathcal{P}$ is any plane, then an equation of $\mathcal{P}$ is any equation with the following properties:

(a) The coordinates (x,y,z) of every point of $\mathcal{P}$ satisfy the equation;

(b) Any values (x,y,z) which satisfy the equation are the coordinates of a point of $\mathcal{P}$.

Consider the following example.

Example 1. Let F be the point $(3,2,4)$ and let $\mathcal{P}$ be the plane containing F and perpendicular to $\overleftrightarrow{OF}$.



A point $R(x,y,z)$ lies in $\mathcal{P}$ if and only if $R = F$ or $\overleftrightarrow{FR} \perp \overleftrightarrow{OF}$. By the Pythagorean Theorem, $\angle OFR$ is a right angle if and only if

$$(OR)^2 = (OF)^2 + (FR)^2 \quad (1)$$

Now $(OR)^2 = (x - 0)^2 + (y - 0)^2 + (z - 0)^2 = x^2 + y^2 + z^2$

and $(OF)^2 = (3 - 0)^2 + (2 - 0)^2 + (4 - 0)^2 = 9 + 4 + 16 = 29$

and $(FR)^2 = (x - 3)^2 + (y - 2)^2 + (z - 4)^2 = x^2 + y^2 + z^2 - 6x - 4y - 8z + 29$.

Hence, (1) becomes

$$x^2 + y^2 + z^2 = 29 + (x^2 + y^2 + z^2 - 6x - 4y - 8z + 29) .$$

Thus

$$6x + 4y + 8z = 2 \cdot 29 ,$$

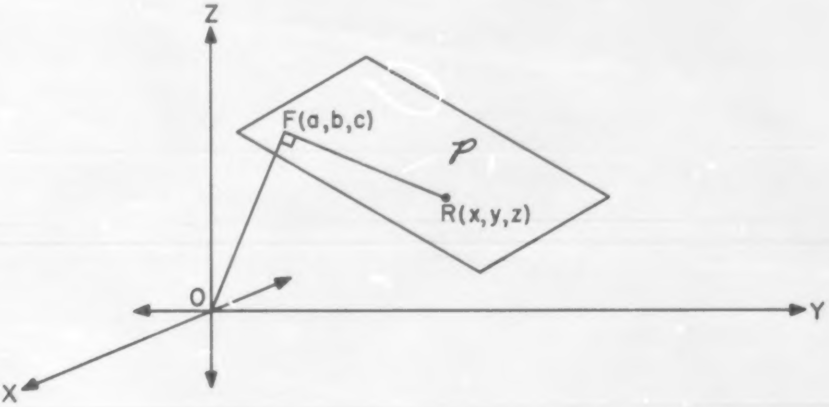
or

$$3x + 2y + 4z = 29 .$$

As a check, we observe that the coordinates $(3,2,4)$ of F satisfy this equation, since $3 \cdot 3 + 2 \cdot 2 + 4 \cdot 4 = 29$.

Note that the numbers $3,2,4$ which appear as the respective coefficients of x,y,z in the equation of the plane are the same as the coordinates of the point F, and that the number 29 is the sum of the squares of the coordinates of F.

We use the discussion in Example 1 as a guide in treating the general case. Suppose that $\mathcal{P}$ is any plane not containing the origin O. Let $F(a,b,c)$ be the point where $\mathcal{P}$ intersects the line containing O and perpendicular to $\mathcal{P}$. By Theorem 9-1, a point $R(x,y,z)$ lies in the plane $\mathcal{P}$ if and only if $R = F$ or $\angle OFR$ is a right angle. But by the Pythagorean Theorem, $\triangle OFR$ has a right angle at F if and only if



$$(OR)^2 = (OF)^2 + (FR)^2 .$$

Since we know the coordinates of O, R, and F, it is a simple matter to obtain the distances OR, OF, and FR by means of the three-dimensional distance formula. Hence the last equation can be written

$$(x - 0)^2 + (y - 0)^2 + (z - 0)^2 = [(a - 0)^2 + (b - 0)^2 + (c - 0)^2] + [(x - a)^2 + (y - b)^2 + (z - c)^2]$$

or, squaring the binomials and collecting terms,

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2 + x^2 - 2ax + a^2 + y^2 - 2by + b^2 + z^2 - 2cz + c^2 ,$$

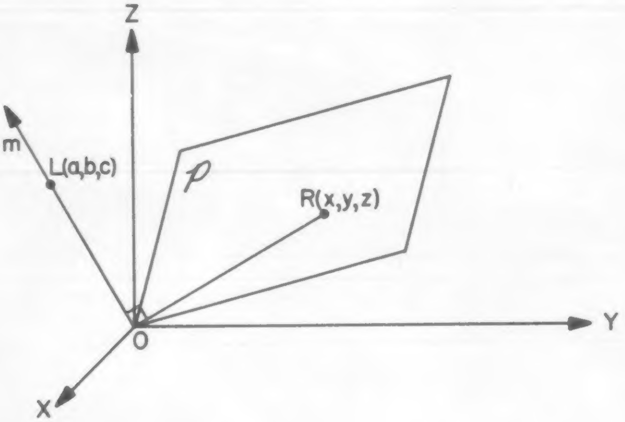
$$2ax + 2by + 2cz = 2a^2 + 2b^2 + 2c^2 ,$$

or finally,

$$ax + by + cz = a^2 + b^2 + c^2 .$$

This equation is satisfied by the coordinates of every point of $\mathcal{P}$ including F , and by the coordinates of no other points. In other words, this equation is an equation of the plane.

If $\mathcal{P}$ contains the origin, the above derivations must be modified a little. In this case, let m be the line which is perpendicular to $\mathcal{P}$ at the origin and let L(a,b,c) be any point on m except the origin. A point R(x,y,z) will now lie in $\mathcal{P}$ if and only if R is the origin or $\angle$LOR is a right angle.



But $\triangle$ LOR will have a right angle at O if and only if

$$(LR)^2 = (LO)^2 + (OR)^2 .$$

Evaluating these distances by means of the three-dimensional distance formula and simplifying as we did in the previous case, we now find that the coordinates (x,y,z) of any point in $\mathcal{P}$, including the origin, must satisfy the equation,

$$ax + by + cz = 0 .$$

The only distinction between this equation and the equation of a plane which does not contain the origin is the value of the constant term.

Our discussion thus far has not touched on the related question: Is every equation of the form

$$ax + by + cz = d$$

an equation of some plane? The answer to this is Yes , but we shall not take time to prove this fact. Instead, we merely summarize our observations in the following theorem.

THEOREM 9-29. Every plane has an equation of the form $ax + by + cz = d$, where one or more of the numbers a, b, c is different from zero; and every equation of this form is an equation of a plane.

Example 2.

What is an equation of the plane which is determined by the points $P_1(2,0,0)$, $P_2(0,1,0)$, $P_3(0,0,3)$?

By Theorem 9-29, we know the required plane has an equation of the form

$$ax + by + cz = d ,$$

which is satisfied, in particular, by the coordinates of P_1 , P_2 , and P_3 . If the coordinates of P_1 satisfy this equation, then substituting,

$$a \cdot 2 + b \cdot 0 + c \cdot 0 = d \quad \text{or} \quad a = \frac{d}{2} .$$

Similarly, since the coordinates of P_2 satisfy the equation,

$$a \cdot 0 + b \cdot 1 + c \cdot 0 = d \quad \text{or} \quad b = d$$

and, since the coordinates of P_3 also satisfy the equation,

$$a \cdot 0 + b \cdot 0 + c \cdot 3 = d \quad \text{or} \quad c = \frac{d}{3} .$$

Substituting for a, b, and c we obtain

$$\frac{d}{2}x + dy + \frac{d}{3}z = d ,$$

or, multiplying both members by 6 and dividing by d ,

$$3x + 6y + 2z = 6 .$$

This is the required equation.

Example 3.

What is an equation of the plane whose points are equidistant from the points $A(1,-3,0)$ and $B(2,0,-5)$?

Let $P(x,y,z)$ be any point. The condition that P lie on the required plane is expressed by the equation $PA = PB$. That is,

$$\sqrt{(x - 1)^2 + (y + 3)^2 + (z - 0)^2} = \sqrt{(x - 2)^2 + (y - 0)^2 + (z + 5)^2}$$

or

$$(x - 1)^2 + (y + 3)^2 + z^2 = (x - 2)^2 + y^2 + (z + 5)^2$$

or

$$(x^2 - 2x + 1) + (y^2 + 6y + 9) + z^2 = (x^2 - 4x + 4) + y^2 + (z^2 + 10z + 25)$$

or (by rearranging terms)

$$(x^2 + y^2 + z^2) - 2x + 6y + 10 = (x^2 + y^2 + z^2) - 4x + 10z + 29 .$$

Finally, by combining terms, we obtain the equation of the plane in simple form:

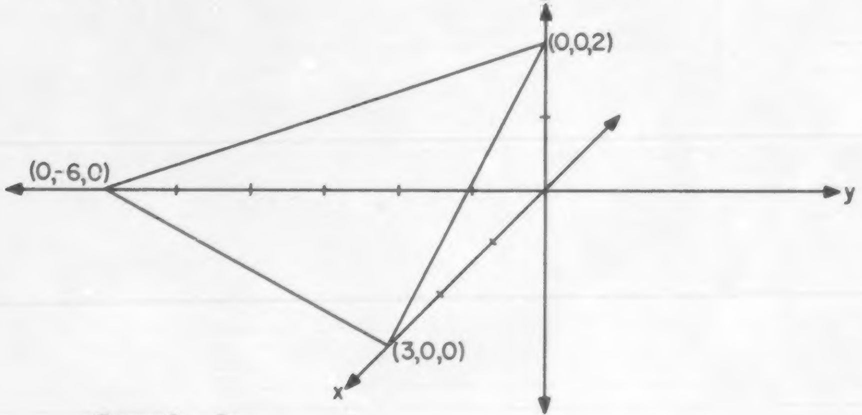
$$2x + 6y - 10z = 19 .$$

Example 4.

Sketch the plane $\mathcal{P}$ given by

$$\mathcal{P} = \{(x,y,z): 2x - y + 3z = 6\} .$$

The coordinates of each point of intersection between $\mathcal{P}$ and a coordinate axis are readily determined, as follows. If a point is on the x-axis, its y-coordinate is zero and its z-coordinate is zero; thus the intersection of $\mathcal{P}$ and the x-axis is the point $(3,0,0)$ because $2x - 0 + 3 \cdot 0 = 6$ yields $x = 3$. Similarly, the y-axis is $\{(x,y,z): x = 0 = z\}$ and it intersects $\mathcal{P}$ at the point $(0,-6,0)$. The point of intersection of $\mathcal{P}$ and the z-axis is $(0,0,2)$. A "sketch" of the plane $\mathcal{P}$ is conveniently made by plotting the triangle whose vertices are the three points $(3,0,0)$, $(0,-6,0)$, $(0,0,2)$.



Example 5.

Find an equation of the plane which contains the three points $(1,2,3)$, $(2,1,-3)$ and $(-1,-2,1)$.

A point in the plane must have coordinates (x,y,z) such that $ax + by + cz = d$. That is, if the point $(1,2,3)$ is in the plane, then

$$a \cdot 1 + b \cdot 2 + c \cdot 3 = d .$$

Similarly, if the point $(2,1,-3)$ is in the plane then,

$$a \cdot 2 + b \cdot 1 + c(-3) = d$$

and, if the point $(-1,-2,1)$ is in the plane then

$$a(-1) + b(-2) + c \cdot 1 = d \ .$$

We find values of $a, b, c,$ in terms of d, which satisfy all three of these equations, to be $a = \frac{11d}{6}$, $b = -\frac{7d}{6}$, $c = \frac{3d}{6}$. Substituting these values in the equation $ax + by + cz = d$ yields

$$\left(\frac{11d}{6}\right)x + \left(-\frac{7d}{6}\right)y + \left(\frac{3d}{6}\right)z = d$$

or

$$11x - 7y + 3z = 6 \ ,$$

an equation of the required plane.

Problem Set 9-9

1. Write an equation of the plane determined by three points whose coordinates are
   (a) $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.
   (b) $(3,0,1)$, $(0,1,0)$, $(0,0,2)$.
   (c) $(3,0,1)$, $(1,2,0)$, $(0,2,4)$.
   (d) $(1,-1,0)$, $(2,0,3)$, $(0,-3,1)$.
2. Determine an equation of the plane whose points are equidistant from $P_1(1,2,3)$ and $P_2(2,5,4)$.
3. Sketch a diagram of the plane represented by each of the equations:
   (a) $5x + 4y = 20$;
   (b) $x + 2y + z = 5$.
4. Find an equation of the plane which contains the point $Q(1,-2,2)$ and is perpendicular to the line containing Q and the origin.

5. Find an equation of the plane which contains the three points whose coordinates are:

   (a) $(0,-2,1)$; $(2,0,-1)$; $(-2,-3,2)$.
   (b) $(1,-2,1)$; $(2,0,-1)$; $(-2,-3,2)$.
   (c) $(1,-2,1)$; $(2,3,-1)$; $(-2,-3,2)$.

6. Find the coordinates of the point of intersection of the plane $\mathcal{P}$ and the line ℓ, if

   $$\mathcal{P} = \{(x,y,z):3x + 5y + 14z = 11\}$$

   and

   $$\ell = \{(x,y,z):x = 2 - 3k\ ,\ y = 1 + k\ ,\ z = 4 - 2k\ ,\ k \text{ real}\}$$

9-10. Summary.

In this chapter we have studied properties of parallelism and perpendicularity for lines and planes. The relationship of parallelism for lines in space is reflexive, symmetric, and transitive. The same three fundamental properties hold for parallelism of planes. The relationship of perpendicularity for lines in space is symmetric, but neither reflexive nor transitive. The same three remarks apply to perpendicularity for planes.

If a point and a line in space are given, there are:

(a) a unique line containing the given point and parallel to the given line,
(b) many planes containing the given point and parallel to the given line,
(c) a unique line containing the given point and perpendicular to the given line,
(d) a unique plane containing the given point and perpendicular to the given line.

If a point and a plane in space are given, there are:

(a) many lines containing the given point and parallel to the given plane,
(b) a unique plane containing the given point and parallel to the given plane,

173

(c) a unique line containing the given point and perpendicular to the given plane,

(d) many planes containing the given point and perpendicular to the given plane.

Given a unit-pair, a postulate in Chapter 3 described for us the distance between two points. In Chapters 4, 5, 6 our theorems on perpendicularity and parallelism enabled us to introduce the distance between a line and a point and the distance between two parallel lines. In Chapter 9 our study of perpendicularity and parallelism permits us to extend the notion of distance again. We can speak of the distance between a point and a plane, the distance between a line and a plane that are parallel to each other, and the distance between two parallel planes.

The ideas of parallelism, perpendicularity, and distance play a basic role in developing a three-dimensional coordinate system. In a one-dimensional system a point is identified by a single real number, in a two-dimensional system by an ordered pair of numbers, and in a three-dimensional system by an ordered triple of numbers. The formula for the distance between two points in space is a natural extension of the formula in two-dimensional geometry. The parametric equations of a line in space are a natural extension of the parametric equations in two-dimensional geometry. The first-degree equation in x, y, z, representing a plane in space, is a natural extension of the first-degree equation in x, y, representing a line in two-dimensional geometry. The coordinate method for proving theorems or analyzing problems is fully as useful and convenient in three-dimensional situations as in two.

VOCABULARY LIST

parallel planes
perpendicular planes
plane angle of a dihedral angle
measure of a dihedral angle
coordinate system (in space)
coordinate plane
distance formula (in space)
equation of a line (in space)
equation of a plane

Review Problems

Chapter 9, Sections 6 to 9

1. Plot the points A, B, C, and D if the coordinates of the points are: A(2,-2,5) ; B(2,3,4) ; C(2,3,-4) ; D(-2,0,4) .

2. Find the distances between the following pairs of points:

   (a) (0,4,5) and (-6,2,8) .
   (b) (3,0,7) and (-1,3,7) .

3. Find the midpoint of each of the segments determined by the pair of points in Problem 2.

4. Write parametric equations of the line determined by each pairs of points in Problem 2.

5. The coordinates of the midpoint of a segment are (8,-4,1) . If the coordinates of one of the endpoints of the segment are (4,-1,3) , find the coordinates of the other endpoint of the segment.

6. The coordinates of the vertices of a triangle are given in each of the following problems. Classify the triangle in each of the problems.

   (a) (5,9,11) , (0,-1,-4) , (5,-11,1) .
   (b) (4,3,-4) , (-2,9,-4) , (-2,3,2) .
   (c) (2,4,2) , (4,5,4) , (4,2,1) .

7. The coordinates of three points are listed in each of the following problems. Tell whether the points are collinear or noncollinear.

   (a) (3,-2,7) , (6,4,-2) , (5,2,1) .
   (b) (1,2,5) , (3,3,2) , (-5,-1,14) .
   (c) (0,4,3) , (1,5,2) , (4,7,0) .
   (d) (3,-1,6) , (1,2,2) , (-1,5,-2) .

8. Given four distinct points A, B, C, and D . If AB = CD and AD = BC , is ABCD a parallelogram? Explain.

9. Given four noncollinear points A, B, C, and D. If the midpoint of $\overline{AC}$ is the midpoint of $\overline{BD}$, is ABCD a parallelogram? Explain.

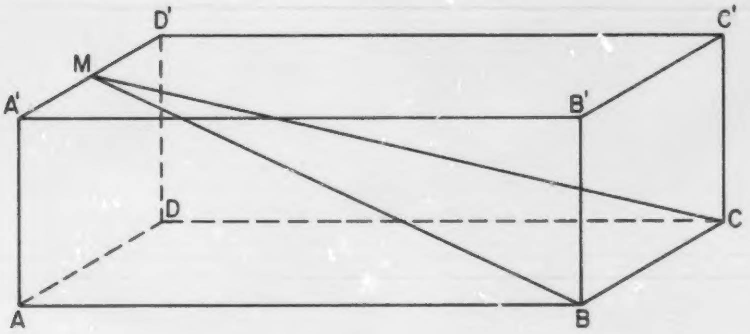
10. Does the plane whose equation is $x + y + z = 4$ contain the point whose coordinates are $(3,-1,2)$?

11. What is the intersection of the xy-plane and the plane whose equation is $2x - 3y + z = 6$?

12. Describe the following: $\{(x,y,z): y = 5\}$.

13. What is the equation of a plane whose points are equidistant from the endpoints of a line segment with coordinates $(-2,-4,7)$ and $(4,5,1)$?

14. M is the midpoint of an edge of the rectangular solid shown in the figure below.
Prove by means of coordinates that $MB = MC$.



REVIEW PROBLEMS

Chapters 6-9

Write + if the statement is true; 0 if the statement is false:

1. The measure of an exterior angle of a triangle is greater than the measure of any interior angle of the triangle.
2. Two antiparallel rays are distinct.
3. The angle opposite the longest side of a triangle is the angle that has the greatest measure.
4. A set of parallel lines intercepts congruent segments on any transversal.
5. If $\overline{AB} \perp \overline{BC}$, then $AB < AC$.
6. There is a triangle whose sides have lengths 351, 513, and 135.
7. Two lines are parallel if each of them is perpendicular to the same line.
8. Given two lines and a transversal of the lines, if one pair of alternate interior angles are congruent, the other pair are also congruent.
9. Given two intersecting lines and a transversal of those lines, no pair of corresponding angles determined by the given transversal are congruent.
10. The bisectors of a pair of consecutive interior angles are parallel.
11. At a point on a line, there are infinitely many lines perpendicular to the line.
12. The distance between a line and a point not on the line is the length of any segment connecting the point and the line.
13. The planes which contain the respective faces of a right dihedral angle are perpendicular.

14. If two angles of one triangle are congruent respectively to two angles of another triangle, then the third angles are congruent.

15. The acute angles of a right triangle are complementary.

16. An exterior angle of a triangle is the supplement of one of the interior angles of the triangle.

17. One of the angles of a right triangle may be an obtuse angle.

18. Two right triangles are congruent if the hypotenuse and a leg of one are congruent respectively to the hypotenuse and a leg of the other.

19. If a line intersects one of two parallel lines, it intersects the other.

20. Two lines that are equal are not parallel.

21. Any two consecutive angles of a parallelogram are supplementary.

22. In $\triangle ABC$, if $m\angle A = 50$ and $m\angle B = 40$, then $\overline{BC}$ is the longest side of the triangle.

23. If x, y, and z are three lines such that $x \parallel y$ and $y \parallel z$, then $x \parallel z$.

24. If x, y, and z are three lines such that $x \perp y$, and $y \perp z$, then $x \perp z$.

25. The contrapositive of a statement is logically equivalent to the converse of the statement.

26. A triangle has a right angle if the lengths of the sides of the triangle are proportional to 7, 24, 25.

27. If one pair of opposite sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

28. The length of the diagonal of a square can be found by multiplying the length of a side by $\sqrt{2}$.

29. A dihedral angle is the union of two halfplanes.

30. Given three distinct coplanar parallel lines and two distinct transversals, the segments formed on one of the transversals are proportional to the corresponding segments formed on the other transversal.

31. In ΔABC, if $m\angle A < m\angle B$, then $AC < BC$.

32. If the lengths of the sides of a triangle are 20, 21, and 31, the triangle is a right triangle.

33. If the measure of one of the angles of a right triangle is 30, then the length of the leg opposite that angle is equal to one-half the length of the hypotenuse.

34. Given a correspondence between two triangles, if two angles of one triangle are congruent to the corresponding angles of the other, the correspondence is a similarity.

35. If a, b, c, are the lengths of the sides of one triangle, if k is a positive number, and if ak, bk, ck, are the lengths of the sides of another triangle, then the triangles are similar.

36. Given two triangles, if an angle of one triangle is congruent to an angle of the other, and two sides of one triangle are proportional to two sides of the other, the triangles are similar.

37. If the legs of a right triangle have lengths a and b, and if the hypotenuse has length c, then $b^2 = (c - a)(c + a)$.

38. Given a correspondence between two triangles, if two angles and a side of one triangle are congruent to the corresponding parts of the other, the correspondence is a congruence.

39. Given a correspondence between two triangles, if two sides and an angle of one triangle are congruent to the corresponding parts of the other, the correspondence is a congruence.

40. Two isosceles triangles are congruent if the vertex angle and the base of one triangle are congruent respectively to the vertex angle and the base of the other.

41. Any two rectangles are similar.

42. If $(a, x) = (x, b)$, then $ab = x^2$.

43. Congruent convex polygons are similar with a proportionality constant of 1.

44. Any two equilateral triangles are similar.

45. If x and y are two distinct positive numbers, if the lengths of two sides of a rectangle are x and y, and if the lengths of the sides of a second rectangle are $x + 6$ and $y + 6$, then the rectangles are similar.

46. If a, b are proportional to c, d with proportionality constant k, and c, d are proportional to e, f with proportionality constant g, then a, b are proportional to e, f with proportionality constant $\frac{k}{g}$.

47. If a line intersects the interiors of two sides of a triangle so that corresponding segments are proportional, the line is parallel to the third side.

48. The ratio of the perimeters of two similar triangles is equivalent to the ratio of any pair of corresponding sides.

49. If p, q are proportional to a, b with proportionality constant k, then $\frac{b}{p} = \frac{a}{q}$.

50. Any real number is permitted to be a constant of proportionality.

51. The lines which contain the respective bisectors of the angles of a triangle are concurrent at a point equidistant from the vertices of the triangle.

52. Any pair of opposite angles of an isosceles trapezoid are supplementary.

53. If the diagonals of a quadrilateral are perpendicular and bisect each other, the quadrilateral is a rhombus.

54. If the coordinates of a quadrilateral ABCD are A(-5, -2), B(-4, 2), C(4, 6), D(3, 1), then the quadrilateral is a parallelogram.

55. The points (0, 6, -2), (3, 4, -2), and (1, 4, -5) are the vertices of an equilateral triangle.

56. If two segments are congruent, their projections on a given line are congruent.

57. If a plane is parallel to one of two distinct parallel lines, it is parallel to the other.

58. Given A(-1, 0), B(6, 2), C(4, 5), D(-3, 3), then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

59. The distance between any two distinct points in a plane is a positive number.

60. The intersection of a line and a plane is a point.

61. If a diagonal of a convex quadrilateral separates it into two congruent triangles, the quadrilateral is a parallelogram.

62. If each pair of opposite sides of a quadrilateral are congruent, the quadrilateral is a parallelogram.

63. The opposite angles of a parallelogram are congruent.

64. A diagonal of a parallelogram bisects two of its angles.

65. The plane whose equation is $x + y + z = 4$ contains the three points (1, 2, 1), (3, -1, 2), and (5, -3, 2).

66. The perimeter of the triangle formed by joining the midpoints of the sides of a given triangle is half the perimeter of the given triangle.

67. If the diagonals of a quadrilateral are perpendicular and congruent, the quadrilateral is a rhombus.

68. If a line intersects one of two parallel planes in a single point, it intersects the other plane in a single point.

69. If the coordinates of points A, B, C are $(5, 9, 11)$, $(0, -1, -4)$, $(5, -11, 1)$, respectively, then ΔABC is a right triangle.

70. $\{(x, y)\colon\ x = 4-4k,\ y = 1-2k,\ k \text{ is real}\}$ contains the point $(3, -1)$.

71. $\{(x, y)\colon\ x = 1 - 3k,\ y = 7k,\ k \text{ is real}\}$ and $\{(x, y)\colon\ x = 9k,\ y = 1 + 21k,\ k \text{ is real}\}$ are parallel lines.

72. $\{(x, y)\colon\ x = 3k,\ y = 3-k,\ k \text{ is real}\}$ is a line passing through the origin.

73. If each of two planes is perpendicular to a third plane, they are parallel to each other.

74. The projection of a line into a plane is a line.

75. If a ray in one face of a dihedral angle is perpendicular to the edge of the dihedral angle, the line containing the ray is perpendicular to the plane containing the other face of the angle.

76. Through a point not in a plane, there is exactly one line perpendicular to the plane.

77. If a plane intersects two other planes in parallel lines, respectively, then the two planes are parallel.

78. If a line is perpendicular to a plane, then any plane containing this line is perpendicular to the given plane.

79. A quadrilateral with three right angles is a rectangle.

80. If $A = (-2, -4, 7)$, $B = (2, 2, 3)$, and $C = (4, 5, 1)$, then A, B, and C are collinear.

81. $\{(x, y, z)\colon\ 2x - 5y + z = 4\}$ contains the point $(3, 2, -1)$.

82. There are infinitely many planes perpendicular to a given line.

83. If a plane is perpendicular to each of two lines, the two lines are coplanar.

84. If each of three noncollinear points of a plane is equidistant from two distinct points P and Q, then PQ is perpendicular to the plane.

85. If each of two planes is parallel to a line, the planes are parallel to each other.

86. If each of two intersecting planes is perpendicular to a third plane, then their line of intersection is perpendicular to this plane.

87. The intersection of the xz-plane and the plane whose equation is $2x - 3y + z = 6$ is $\{(x, y, z): 2x + z = 6 \text{ and } y = 0\}$.

88. If A, B, C, and D are four distinct points in space such that $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$, then ABCD is a parallelogram.

89. $\{(x, y): x = 3k, y = 2k, 0 \leq k \leq 1\}$ is a segment.

90. $\{(x, y): x = 2 + 3k, y = 2k, k \text{ is real}\}$ is a line having the slope $\frac{3}{2}$.

91. $\{(x, y): x = 3 + k, y = 1 + 2k, k \geq 0\}$ is contained in Quadrant I.

92. $\{(x, y): x = 3k, y = 2k, k \text{ is real}\}$ and $\{(x, y): x = 3\}$ intersect at the point (3, 2).

93. The intersection of $\{(x, y): x = 3k, y = 2k, k \text{ is real}\}$ and $\{(x, y): x = 2h, y = 3h, h \text{ is real}\}$ is the origin.

94. $\{(x, y): x = 3k, y = 2k, k \text{ is real}\}$ is perpendicular to $\{(x, y): x = 2k, y = 3k, k \text{ is real}\}$.

95. If two angles have the property that the sides of one are antiparallel to the corresponding sides of the other, the angles are supplementary.

96. If ABC is a right triangle and $\overline{CD}$ is the altitude to the hypotenuse of the triangle, then ΔABC is similar to ΔACD.

97. The union of the set of all rhombuses and the set of all rectangles is the set of all squares.

98. If ABCD is a quadrilateral and $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$, then ABCD is a parallelogram.

99. If ABC is a right triangle and $\overline{CD}$ is the altitude to the hypotenuse of the triangle, then the square of the length of $\overline{CD}$ is equal to the product of the length of $\overline{AD}$ and the length of $\overline{DB}$.

100. The altitude of an equilateral triangle each of whose sides has length s is $\frac{s}{2}\sqrt{2}$.

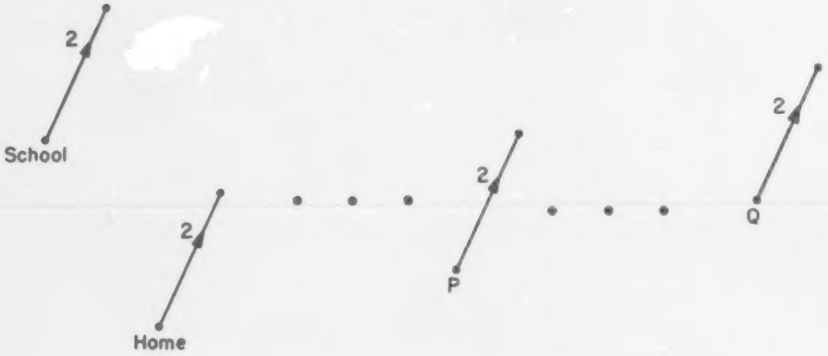
Chapter 10

DIRECTED SEGMENTS AND VECTORS

10-1. Introduction.

Many of the quantities which we encounter in life are adequately described by a unit of measure and a number. Expressions of this sort, such as 5 inches, 20 degrees, 75 cents, 2 hours, 10 cubic feet, or 15 miles per gallon, can be represented as the distance between two points on the appropriate scale, and for this reason are called scalars. On the other hand, there are numerous quantities such as displacement, velocity, acceleration, and force, for which more information than this must be given before they are adequately specified.

Consider the simple idea of a displacement, for instance. If we are told that a boy walked two miles, we really don't know very much about what he did unless we are also told the direction in which he walked. Even if we know that he walked two miles northeast, say, we don't have a complete description of what he did. He might have started at school and walked two miles northeast, or he might have started at his home, or any number of other points:



We have an adequate description of the boy's walk only if we know

(a) the point from which he started,
(b) the direction in which he walked,
(c) the distance he walked;

or, equivalently, if we know

(a) the point from which he started,
(b) the point at which he ended.

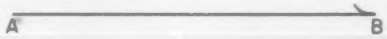
Speaking in somewhat more abstract terms, it appears that a displacement can be represented in either of two ways:

(a) By a segment extending a given distance in a given direction from a given point.
(b) By a pair of points, one identified as a starting point, the other as a terminal point.

A segment, as we defined it in Section 3-6, is just a set of points and has no direction associated with it. Similarly, the set consisting of the endpoints of a segment has no direction associated with it. Hence neither the segment $\overline{AB}$ nor the set $\{A,B\}$ is adequate to describe the displacement from A to B because neither distinguishes between this displacement and the displacement from B to A, which is quite a different thing. Clearly, we can specify the displacement which starts at the point A and ends at the point B by using the *ordered pair of points* (A,B). However, if we wish to describe a displacement by means of a segment we must extend our original definition:

> *DEFINITION*. A segment is a *directed segment* if and only if one of its endpoints is designated as its *initial point*, or *origin*, and the other endpoint is designated as its *terminal point* or *terminus*.

Notation. The symbol $(\overrightarrow{A,B})$ is used to denote the directed segment whose origin is A and whose terminus is B . In a drawing a directed segment is shown by placing a half arrow-head at its terminal point, thus:



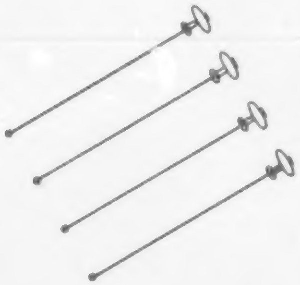
DEFINITION. Directed segments which have the same initial points and the same terminal points are said to be equal.

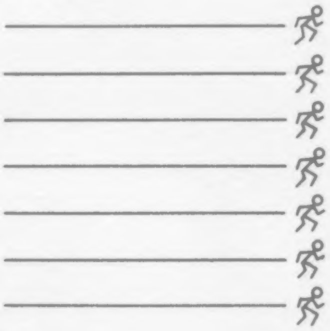
DEFINITION. By the length of the directed segment $(\overrightarrow{A,B})$ we mean AB , that is, the length of the associated segment $\overline{AB}$.

In the next section we develop some of the properties of directed segments, using the concept of displacement to motivate our work. Then in later sections we introduce the important generalization of a directed segment known as a vector.

10-2. Directed Segments.

In the last section we introduced the idea of a directed segment by considering the displacement of a single object from one point to another. Let us now consider a number of objects which move equal distances in the same direction along parallel lines, as for example a group of planes flying in formation or the linemen of a football team charging down the field together in their pregame warm-up:

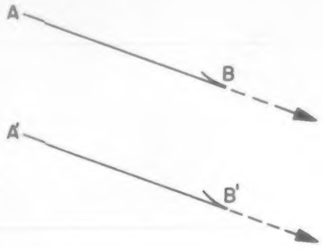


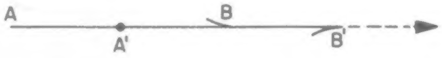


The displacement of the planes, in the first case, and of the players in the second, are all different because no two begin and end at the same points. None the less, in each of the two cases there are characteristics common to all the displacements; specifically, the displacements take place in the same direction along parallel lines and involve movement through equal distances In many applications it is convenient to be able to refer concisely to directed segments with these characteristics, and to provide for this we introduce the following definition:

> DEFINITION. Two directed segments, $(\overrightarrow{A,B})$ and $(\overrightarrow{C,D})$ are equivalent if and only if $AB = CD$ and $\overrightarrow{AB} \parallel \overrightarrow{CD}$.

(We suggest that you review Section 6-7 regarding parallel rays at this point.) This definition is illustrated in the following figures:





In each case the directed segments $(\overrightarrow{A,B})$ and $(\overrightarrow{A',B'})$ are equivalent because $AB = A'B'$ and $\overrightarrow{AB} \parallel \overrightarrow{A'B'}$. In these figures, are the directed segments $(\overrightarrow{A,B})$ and $(\overrightarrow{B',A'})$ equivalent? Why?

> Notation. We indicate that $(\overrightarrow{A,B})$ is equivalent to $(\overrightarrow{C,D})$ by writing
>
> $$(\overrightarrow{A,B}) \doteq (\overrightarrow{C,D}) .$$

If you understand the definition of equivalent directed segments you should have no trouble verifying the following statements.

Properties of Directed Segment Equivalence.

1. Directed segment equivalence is reflexive:

$$(\overrightarrow{A,B}) \doteq (\overrightarrow{A,B}) \text{ .}$$

2. Directed segment equivalence is symmetric:

If $(\overrightarrow{A,B}) \doteq (\overrightarrow{C,D})$, then $(\overrightarrow{C,D}) \doteq (\overrightarrow{A,B})$.

3. Directed segment equivalence is transitive:

If $(\overrightarrow{A,B}) \doteq (\overrightarrow{C,D})$ and $(\overrightarrow{C,D}) \doteq (\overrightarrow{E,F})$,

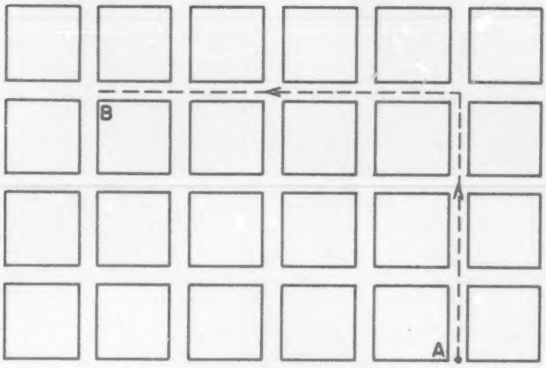
then $(\overrightarrow{A,B}) \doteq (\overrightarrow{E,F})$.

A fundamental property of directed segments is given by the following theorem:

THEOREM 10-1. There is one and only one directed segment which is equivalent to a given directed segment and has its origin at a given point.

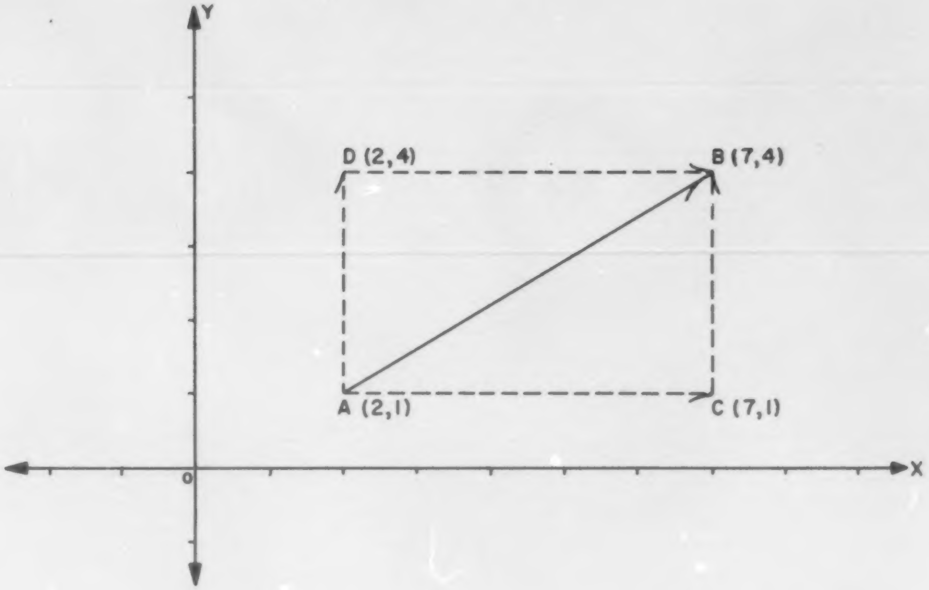
Proof: By definition, the directed segment which has its origin at P and is equivalent to $(\overrightarrow{A,B})$ must lie on the unique line ℓ , which contains P and is parallel to $\overleftrightarrow{AB}$. Moreover, the required directed segment must lie on the unique ray of ℓ which has P for its endpoint and is parallel to $\overrightarrow{AB}$. Finally, on this ray, the terminal point, Q , of the required directed segment must have the property that $AB = PQ$, and by the Point Plotting Theorem, there is one and only one such point. Hence the theorem is proved.

If we return to the idea of displacement for a moment, and consider the problem of getting from one point to another in a city, it is clear that only rarely can one go directly from one point to another. Usually, because of the buildings which are in the way, one must walk down one street a certain distance, then turn a corner, and continue on another street to reach his destination.



Instructions for getting around in a city reflect this fact, and we are all accustomed to being told to go so many blocks in one direction and then go so many blocks in another to get to some desired address.

The observations suggest, first of all, that any displacement can be achieved in various ways by a succession of other displacements. In the second place, they suggest that it may be convenient when speaking of displacements to use a coordinate system in describing simple displacements which together produce a given displacement. For instance, it is clear from the following figure that the displacement from the point A(2,1) to the point B(7,4) can be achieved by first performing the displacement from the initial point A(2,1) to



the point $C(7,1)$ and then continuing with the displacement from C to the terminal point $B(7,4)$. It can also be accomplished by first performing the displacement from the initial point A to the point $D(2,4)$ and then continuing with the displacement from D to the terminal point B. A displacement to the right of $7 - 2 = 5$ and an upward displacement of $4 - 1 = 3$, regardless of the order in which they are performed, thus combine to give precisely the displacement from A to B. In terms of the given coordinate system, the numbers 5 and 3 associated with the displacements parallel to ray $\overrightarrow{OX}$ and the ray $\overrightarrow{OY}$, respectively, are thus fundamentally related to the displacement from $A(2,1)$ to $B(7,4)$.

The extension of the foregoing observations to directed segments is contained in the following definitions.

DEFINITIONS. If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are points in the xy-plane, the number $x_2 - x_1$ is called the *x-component* of $(\overrightarrow{P_1,P_2})$ and the number $y_2 - y_1$ is called the *y-component* of $(\overrightarrow{P_1,P_2})$.

The ordered pair of numbers $[x_2 - x_1 \, , \, y_2 - y_1]$ are called the *components* of $(\overrightarrow{P_1,P_2})$. (Note the use of brackets to indicate components.)

Although a given directed segment determines a unique pair of components, it is not true that a given pair of components determines a unique directed segment. For instance, as we have just seen, the points $A(2,1)$ and $B(7,4)$ determine a directed segment with components $[5,3]$, but so do the points $C(4,2)$ and $D(9,5)$. What is another directed segment with components $[5,3]$? There are many more, but it is possible to derive the following theorem.

THEOREM 10-2. Two directed segments $(\overrightarrow{P_1,P_2})$ and $(\overrightarrow{P_3,P_4})$ are equivalent if and only if they have the same components.

To establish this theorem, we must show two things:

1. If $(\overrightarrow{P_1,P_2})$ and $(\overrightarrow{P_3,P_4})$ have the same components, they are equivalent.
2. If $(\overrightarrow{P_1,P_2})$ and $(\overrightarrow{P_3,P_4})$ are equivalent, they have the same components.

To prove these assertions requires the consideration of two cases, according as the four points P_1 , P_2 , P_3 , P_4 are or are not collinear. To save time, we shall give only the outline of the proof in the general case where the points are noncollinear.

1. Let the coordinates of P_1, P_2, P_3, P_4 be, respectively, (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4) . Then the hypothesis of the first part of the theorem is that

$$x_2 - x_1 = x_4 - x_3 \quad \text{and} \quad y_2 - y_1 = y_4 - y_3 \ .$$

If $x_2 - x_1 = x_4 - x_3 = 0$, then the lines $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ are vertical and hence parallel. If $x_2 - x_1 = x_4 - x_3 \neq 0$, then by division

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3} \ .$$

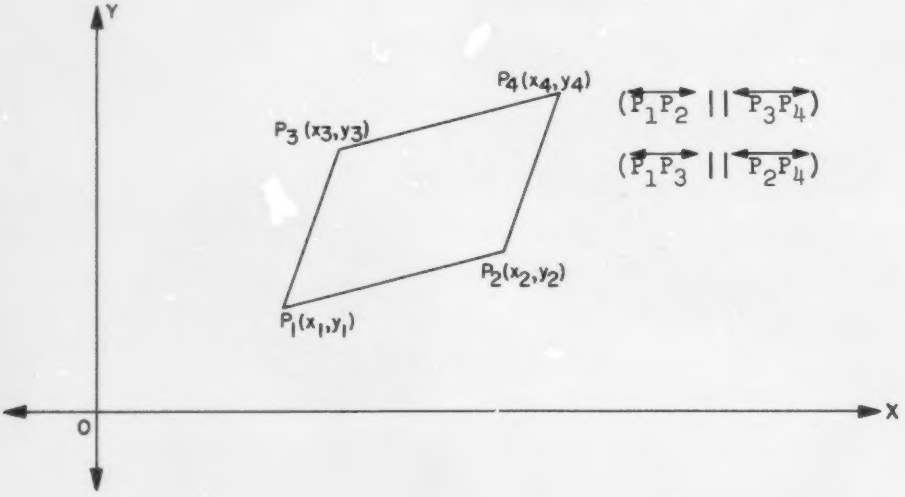
Therefore, the slopes of $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ are equal and hence, by Theorem 8-12, the lines are parallel. In every case, then, when the given points are not collinear, $\overleftrightarrow{P_1P_2}$ is parallel to $\overleftrightarrow{P_3P_4}$. Now from the hypothesis that

$$x_2 - x_1 = x_4 - x_3 \quad \text{and} \quad y_2 - y_1 = y_4 - y_3 \ ,$$

we obtain at once

$$x_3 - x_1 = x_4 - x_2 \quad \text{and} \quad y_3 - y_1 = y_4 - y_2 \ .$$

Hence, by an argument analogous to the one we have just given, it follows that $\overleftrightarrow{P_1P_3}$ is parallel to $\overleftrightarrow{P_2P_4}$.



$(\overleftrightarrow{P_1P_2} \,||\, \overleftrightarrow{P_3P_4})$

$(\overleftrightarrow{P_1P_3} \,||\, \overleftrightarrow{P_2P_4})$

Therefore, if P_1, P_2, P_3, P_4 are not collinear, they form a parallelogram, $P_1P_2P_3P_4$. Hence

$$P_1P_2 = P_3P_4 ,$$

and P_2 and P_4 lie on the same side of the line $\overleftrightarrow{P_1P_3}$. Therefore $(\overrightarrow{P_1,P_2})$ and $(\overrightarrow{P_3,P_4})$ are equivalent, as asserted.

2. Now suppose that $(\overrightarrow{P_1,P_2}) \doteq (\overrightarrow{P_3,P_4})$. Let $P_5 = (x_2 - x_1 + x_3 , y_2 - y_1 + y_3)$. Then the components of $(\overrightarrow{P_3,P_5})$ are

$$[(x_2 - x_1 + x_3) - x_3 , (y_2 - y_1 + y_3) - y_3]$$

or

$$[x_2 - x_1 , y_2 - y_1] .$$

But these are also the components of $(\overrightarrow{P_1,P_2})$. It follows, then, from the first part of this theorem (which we have proved) that

$$(\overrightarrow{P_3,P_5}) \doteq (\overrightarrow{P_1,P_2}) .$$

But the hypothesis of this part tells us that

$$(\overrightarrow{P_1,P_2}) \doteq (\overrightarrow{P_3,P_4}) .$$

Because of the transitivity property of directed segment equivalence we conclude that

$$(\overrightarrow{P_3,P_5}) \doteq (\overrightarrow{P_3,P_4}) .$$

By Theorem 10-1, it follows that $P_5 = P_4$, or

$$x_2 - x_1 + x_3 = x_4 \text{ and } y_2 - y_1 + y_3 = y_4 ,$$

or

$$x_2 - x_1 = x_4 - x_3 \text{ and } y_2 - y_1 = y_4 - y_3 .$$

This completes the proof of the theorem.

As an immediate consequence of this last theorem, together with the Pythagorean Theorem, we have the following result:

THEOREM 10-3. If P_1 and P_2 have coordinates (x_1,y_1) and (x_2,y_2), respectively, the length of any directed segment equivalent to $(\overrightarrow{P_1,P_2})$ is

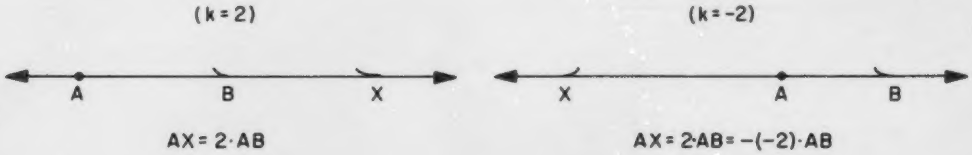
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

We have, in effect, defined equivalent directed segments as directed segments having the same length and direction, regardless of their initial points. Another interesting class of directed segments consists of those having the same initial point and direction, regardless of their lengths. This leads to the idea of the product of a directed segment and a number, which is made precise in the following definition:

DEFINITION. Let $(\overrightarrow{A,B})$ be any directed segment and let k be any real number. The product, $k(\overrightarrow{A,B})$, is the directed segment $(\overrightarrow{A,X})$ where X is the point whose coordinate is k in the coordinate system on $\overleftrightarrow{AB}$, with origin A and unit point B.

DEFINITION. The directed segment $-1 \cdot (\overrightarrow{A,B}) = -(\overrightarrow{A,B})$ is called the opposite of the directed segment $(\overrightarrow{A,B})$.

The following figure illustrates the multiplication of a directed segment by the numbers 2 and -2:



We note that, if $k > 0$ then X is in $\overrightarrow{AB}$; if $k < 0$, then X is in the ray opposite to $\overrightarrow{AB}$; if $k = 0$, then $X = A$. This last result introduces the possibility of $(\overrightarrow{A,A})$, a zero directed segment.

As we might expect, when a directed segment is multiplied by a number, k, the components of the directed segment are both multiplied by the same number. More precisely we have the following theorem.

THEOREM 10-4. If the coordinates of P_1 and P_2 are (x_1,y_1) and (x_2,y_2), respectively, then the components of the directed segment $(\overrightarrow{P_1,P_3})$ which is k times the directed segment $(\overrightarrow{P_1,P_2})$ are $k(x_2 - x_1)$ and $k(y_2 - y_1)$.

Proof: Let $(\overrightarrow{P_1,P_3}) \doteq [x_3 - x_1\ ,\ y_3 - y_1]$.

By the Two Point Theorem

$$x_3 = x_1 + k(x_2 - x_1) \quad \text{and} \quad y_3 = y_1 + k(y_2 - y_1)$$

or

$$x_3 - x_1 = k(x_2 - x_1) \quad \text{and} \quad y_3 - y_1 = k(y_2 - y_1)\ .$$

Thus $$(\overrightarrow{P_1,P_3}) = [k(x_2 - x_1)\ ,\ k(y_2 - y_1)]\ .$$

Problem Set 10-2a

1. A and B are two points. List all the directed line segments they determine.
2. A, B, C are three points. List all the directed line segments they determine.
3. In the figures below list the equivalent directed line segments.

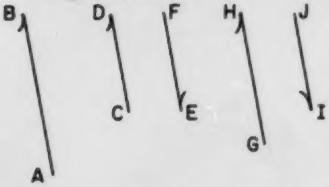


FIGURE a

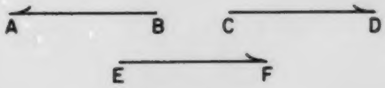


FIGURE b

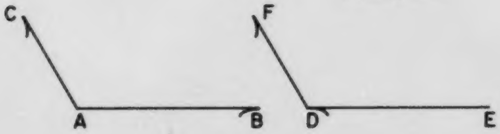
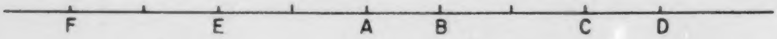


FIGURE c

4. If A, B, C, D are distinct points and $(\overrightarrow{A,B}) \doteq (\overrightarrow{C,D})$, show for each of the following cases that $(\overrightarrow{A,C}) \doteq (\overrightarrow{B,D})$:
   - (a) A, B, C, D are collinear in that order.
   - (b) A, C, B, D are collinear in that order.
   - (c) No three of A, B, C, D are collinear.
   - (d) Why did we not consider three points collinear in Part (c) ?
   - (e) Why do we not consider A, C, D, B collinear in that order?

5. If B, F, G, H are distinct points, $(\overrightarrow{B,F}) \doteq (\overrightarrow{G,H})$, and a line ℓ is not perpendicular to $\overleftrightarrow{BF}$ and does not intersect $\overline{BF}$, show that the projections of $(\overrightarrow{B,F})$ and $(\overrightarrow{G,H})$ on ℓ are equivalent directed segments. Let the projection of B on ℓ be B', and consider three cases:
   - (a) G is contained in $\overleftrightarrow{BB'}$.
   - (b) G is in the same halfplane as F with respect to $\overleftrightarrow{BB'}$.
   - (c) G is in a different halfplane from F with respect to $\overleftrightarrow{BB'}$.

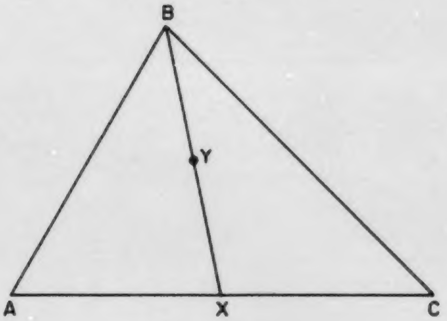
6. Determine k so that each of the following statements is true.



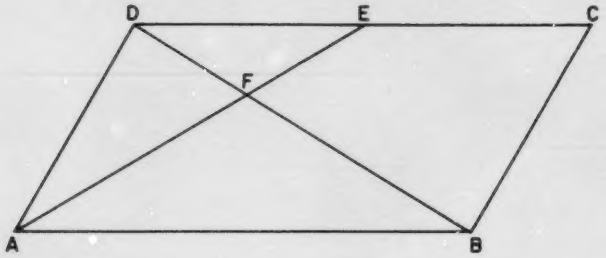
(a) $(\overrightarrow{A,C}) \doteq k(\overrightarrow{A,B})$.
(b) $(\overrightarrow{A,E}) \doteq k(\overrightarrow{A,B})$.
(c) $(\overrightarrow{A,F}) \doteq k(\overrightarrow{A,E})$.
(d) $(\overrightarrow{D,A}) \doteq k(\overrightarrow{A,F})$.
(e) $(\overrightarrow{A,E}) \doteq k(\overrightarrow{A,C})$.
(f) $(\overrightarrow{A,D}) \doteq k(\overrightarrow{A,F})$.
(g) $(\overrightarrow{A,F}) \doteq k(\overrightarrow{A,B})$.
(h) $(\overrightarrow{B,C}) \doteq k(\overrightarrow{A,D})$.
(i) $(\overrightarrow{B,C}) \doteq k(\overrightarrow{A,F})$.

7. A, B, X are collinear points. Find r such that $(\overrightarrow{A,X}) \doteq r(\overrightarrow{A,B})$ and s such that $(\overrightarrow{B,X}) = s(\overrightarrow{B,A})$, if
   - (a) X is the midpoint of segment $\overline{AB}$;
   - (b) B is the midpoint of segment $\overline{AX}$;
   - (c) A is the midpoint of segment $\overline{BX}$;
   - (d) X is two-thirds of the way from A to B ;
   - (e) B is two-thirds of the way from A to X ;
   - (f) A is two-thirds of the way from B to X .

8. In triangle ABC , X is the midpoint of $\overline{AC}$ and Y is the midpoint of $\overline{BX}$. Determine k so that each of the following statements is true.
   - (a) $(\overrightarrow{B,X}) \doteq k(\overrightarrow{B,Y})$.
   - (b) $(\overrightarrow{B,Y}) \doteq k(\overrightarrow{B,X})$,
   - (c) $(\overrightarrow{A,C}) \doteq k(\overrightarrow{C,A})$.
   - (d) $(\overrightarrow{A,C}) \doteq k(\overrightarrow{C,X})$.
   - (e) $(\overrightarrow{C,X}) \doteq k(\overrightarrow{X,A})$.
   - (f) $(\overrightarrow{C,A}) \doteq k(\overrightarrow{A,X})$.
   - (g) $(\overrightarrow{X,B}) \doteq k(\overrightarrow{B,Y})$.



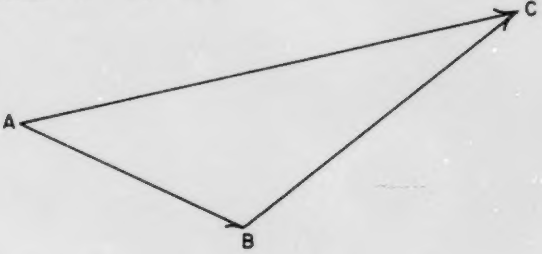
9. In the parallelogram ABCD , E is the midpoint of $\overline{CD}$ and $\overline{AE}$ trisects $\overline{BD}$ at F as indicated below.



Determine k so each of the following statements is true.

- (a) $(\overrightarrow{D,F}) \doteq k(\overrightarrow{D,B})$.
- (b) $(\overrightarrow{D,E}) \doteq k(\overrightarrow{C,D})$.
- (c) $(\overrightarrow{B,D}) \doteq k(\overrightarrow{B,F})$.
- (d) $(\overrightarrow{F,B}) \doteq k(\overrightarrow{D,B})$.
- (e) $(\overrightarrow{F,B}) \doteq k(\overrightarrow{B,D})$.
- (f) $(\overrightarrow{A,B}) \doteq k(\overrightarrow{D,C})$.
- ([illegible]) $(\overrightarrow{A,D}) \doteq k(\overrightarrow{C,B})$.

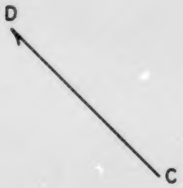
Having now discussed briefly the product of a directed segment and a number, it is natural to ask if the sum of two directed segments can be defined. Since the displacement from B to C following, or in a sense "added to," the displacement from A to B accomplishes exactly the same thing as the displacement from A to C ,

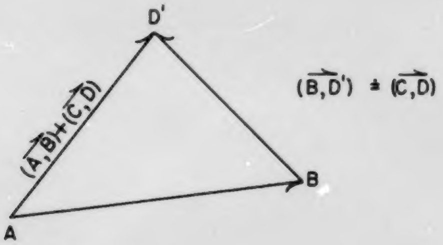


it seems natural to write

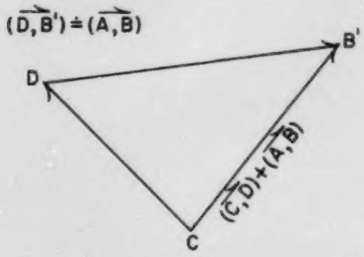
$$(\overrightarrow{A,B}) + (\overrightarrow{B,C}) = (\overrightarrow{A,C}) .$$

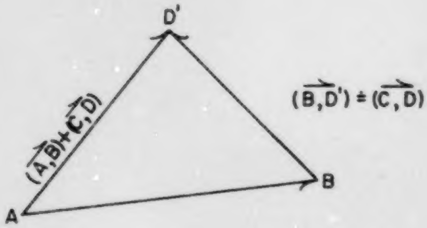
However, as a possible definition of the sum of two directed segments, this expression has serious limitations, for it pertains only to two directed segments for which the terminus of the first is the origin of the second. With the idea of equivalent directed segments in mind, we might go further and say that to add $(\overrightarrow{C,D})$ to $(\overrightarrow{A,B})$ where B and C are different points, first determine the unique directed segment, $(\overrightarrow{B,D'})$, which is equivalent to $(\overrightarrow{C,D})$ (Theorem 10-1) and then add it to $(\overrightarrow{A,B})$, according to the above geometric definitions.




D'
$(\overrightarrow{A,B})+(\overrightarrow{C,D})$
$(\overrightarrow{B,D'}) \doteq (\overrightarrow{C,D})$
B
A

This procedure permits us to form the sum of any two directed segments. However, the resulting process is a very curious type of addition, for, as the following figure shows, $(\overrightarrow{A,B}) + (\overrightarrow{C,D})$ is not the same as $(\overrightarrow{C,D}) + (\overrightarrow{A,B})$.

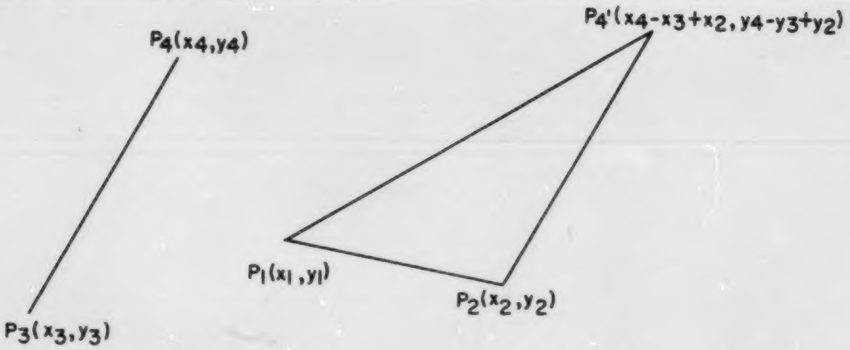

$(\overrightarrow{D,B'}) \doteq (\overrightarrow{A,B})$
B'
D
$(\overrightarrow{C,D}) + (\overrightarrow{A,B})$
C


D'
$(\overrightarrow{B,D'}) \doteq (\overrightarrow{C,D})$
$(\overrightarrow{A,B}) + (\overrightarrow{C,D})$
B
A

From the preceding figure, it appears that although $(\overrightarrow{A,B}) + (\overrightarrow{C,D})$ is not the same as $(\overrightarrow{C,D}) + (\overrightarrow{A,B})$, these two directed segments are equivalent. That this is actually the case follows from the next theorem.

THEOREM 10-5. The components of $(\overrightarrow{P_1,P_2}) + (\overrightarrow{P_3,P_4})$ are the sums of the corresponding components of $(\overrightarrow{P_1,P_2})$ and $(\overrightarrow{P_3,P_4})$.

To prove this, let the coordinates of P_1, P_2, P_3, P_4 be (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4) , respectively. Then the components of $(\overrightarrow{P_1,P_2})$ are $[x_2 - x_1 , y_2 - y_1]$ and the components of $(\overrightarrow{P_3,P_4})$ are $[x_4 - x_3 , y_4 - y_3]$. Now to add $(\overrightarrow{P_3,P_4})$ to $(\overrightarrow{P_1,P_2})$ we must first determine the directed segment, (P_2,P_4') which is equivalent to (P_3,P_4) . Since equivalent directed segments have the same components (Theorem 10-2) it follows that the components of $(\overrightarrow{P_2,P_4'})$ are



$$x_4' - x_2 = x_4 - x_3 \quad \text{and} \quad y_4' - y_2 = y_4 - y_3 .$$

Hence the coordinates of P_4' are

$$x_4' = x_4 - x_3 + x_2 \quad \text{and} \quad y_4' = y_4 - y_3 + y_2 .$$

Therefore the components of the sum

$(\overrightarrow{P_1,P_2}) + (\overrightarrow{P_3,P_4}) = (\overrightarrow{P_1,P_4'})$ are

$$(x_4 - x_3 + x_2) - x_1 = (x_4 - x_3) + (x_2 - x_1)$$

and

$$(y_4 - y_3 + y_2) - y_1 = (y_4 - y_3) + (y_2 - y_1) .$$

Clearly, the components of the sum, $(\overrightarrow{P_1,P_2}) + (\overrightarrow{P_3,P_4})$ are the sums of the corresponding components of $(\overrightarrow{P_1,P_2})$ and $(\overrightarrow{P_3,P_4})$, as asserted.

From this theorem, it is apparent that $(\overrightarrow{P_1,P_2}) + (\overrightarrow{P_3,P_4})$ and $(\overrightarrow{P_3,P_4}) + (\overrightarrow{P_1,P_2})$ have the same components. Hence, by Theorem 10-2 they are equivalent. Thus we have the following theorem:

THEOREM 10-6. $(\overrightarrow{P_1,P_2}) + (\overrightarrow{P_3,P_4})$ and $(\overrightarrow{P_3,P_4}) + (\overrightarrow{P_1,P_2})$ are equivalent directed segments.

The following theorem is also an immediate consequence of Theorem 10-5.

THEOREM 10-7. If $(\overrightarrow{P_1,P_2}) \doteq (\overrightarrow{Q_1,Q_2})$ and $(\overrightarrow{P_3,P_4}) \doteq (\overrightarrow{Q_3,Q_4})$ then $(\overrightarrow{P_1,P_2}) + (\overrightarrow{P_3,P_4}) \doteq (\overrightarrow{Q_1,Q_2}) + (\overrightarrow{Q_3,Q_4})$.

On the basis of the preceding discussion, it should occur to us that instead of focusing our attention on directed segments, it might be better to consider as fundamental entities the various *sets* of equivalent directed segments. This is really not difficult to do, even though each set contains infinitely many members, for according to Theorem 10-2, each such set is characterized by a unique pair of components, and conversely. In other words, there is a one-to-one correspondence which matches each ordered pair of real numbers with each set of equivalent directed segments. Moreover, if we define the sum of two sets of equivalent directed segments, S_1 and S_2 , to be the unique set which contains the sum of any directed segment from S_1 and any directed segment from S_2 , we have a process of addition in which, by Theorem 10-5 and Theorem 10-6, it is true that

$$S_1 + S_2 = S_2 + S_1 .$$

In the rest of this chapter we shall adopt the point of view we have just described. Sets of equivalent directed segments, or the ordered pairs of components which are in one-to-one correspondence with these sets, we shall call *vectors*. A directed segment is thus *not* a vector, although it clearly *determines* the vector consisting of all the directed segments equivalent to the given one. Each directed segment is thus a representation of a vector, in somewhat the same way that each member of a set of equivalent fractions such as

$$\frac{1}{2} , \frac{2}{4} , \frac{5}{10} , \frac{9}{18} , \ldots$$

is a representation of a unique real number.

In the next section we shall introduce formal definitions of vectors and their properties. However, these are all motivated by the properties of directed segments which we have discussed in this section. If you keep the latter in mind, the work ahead of you should seem a natural extension of what we have already done.

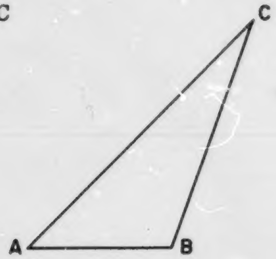
Problem Set 10-2b

1. You may recall from the algebra of real numbers the following definition:

   DEFINITION. If a, b are two real numbers, then $a - b$ is the real number c such that $b + c = a$. The operation of finding c where a, b are given is subtraction.

   Using the above definition as a guide write a definition for the subtraction of two directed line segments.

2. In triangle ABC



(a) $(\overrightarrow{A,B}) + (\overrightarrow{B,C}) \doteq \ ?$

(b) $(\overrightarrow{B,A}) + \ ? \ \doteq (\overrightarrow{B,C})$.

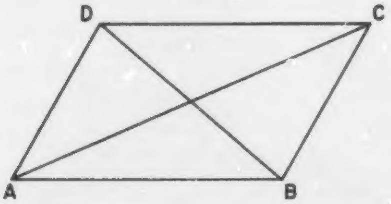
(c) $? \ + (\overrightarrow{B,A}) \doteq (\overrightarrow{B,C})$.

(d) $? \ + (\overrightarrow{A,B}) \doteq (\overrightarrow{A,A})$.

(e) $(\overrightarrow{A,B}) + (\overrightarrow{B,C}) + (\overrightarrow{C,A}) \doteq \ ?$

(f) $(\overrightarrow{B,A}) + (\overrightarrow{A,C}) + (\overrightarrow{C,B}) \doteq \ ?$

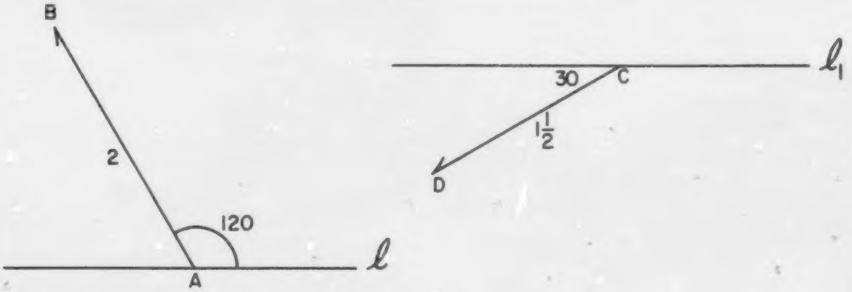
(g) $(\overrightarrow{C,A}) + \ ? \ \doteq (\overrightarrow{C,B})$.



3. In parallelogram ABCD

(a) $(\overrightarrow{A,B}) \doteq$?
(b) $(\overrightarrow{A,D}) \doteq$?
(c) $(\overrightarrow{A,B}) + (\overrightarrow{B,C}) + (\overrightarrow{C,D}) \doteq$?
(d) $(\overrightarrow{A,D}) + (\overrightarrow{D,B}) + (\overrightarrow{B,A}) \doteq$?

4. Given two directed line segments $(\overrightarrow{A,B})$, $(\overrightarrow{C,D})$ and horizontal lines ℓ and ℓ_1, as indicated below.

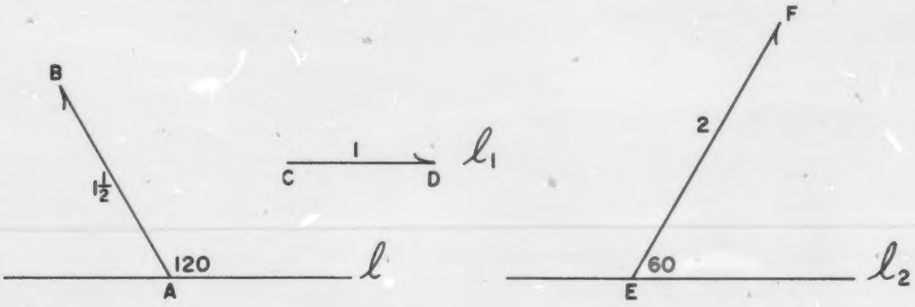


Find graphically

(a) $(\overrightarrow{A,B}) + (\overrightarrow{C,D})$.
(b) $(\overrightarrow{C,D}) + (\overrightarrow{A,B})$.

What is true about the two sums?

5. Given $(\overrightarrow{A,B})$, $(\overrightarrow{C,D})$, $(\overrightarrow{E,F})$ as shown below; ℓ, ℓ_1, ℓ_2 are horizontal lines.



Find graphically

(a) $(\overrightarrow{A,B}) + (\overrightarrow{C,D})$.
(b) $(\overrightarrow{C,D}) + (\overrightarrow{E,F})$.
(c) $(\overrightarrow{A,B}) + (\overrightarrow{E,F})$.

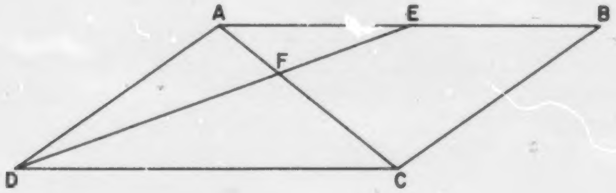
(d) $(\overrightarrow{A,B}) + \left((\overrightarrow{C,D}) + (\overrightarrow{E,F})\right)$. Do this by adding the sum in (b) to $(\overrightarrow{A,B})$.

(e) $\left((\overrightarrow{A,B}) + (\overrightarrow{C,D})\right) + (\overrightarrow{E,F})$. Do this by adding $(\overrightarrow{E,F})$ to the sum in (a).

What appears to be true of the two sums in (d) and (e)?

6. Letting 1 inch represent 2 miles, find graphically the resultant displacement if an automobile travels 4 miles due north then 5 miles northeast.

7. ABCD is a parallelogram. E is the midpoint of $\overline{AB}$; $\overline{AC}$ and $\overline{DE}$ trisect each other at F .



(a) $(\overrightarrow{D,A}) + (\overrightarrow{A,E}) \doteq$?

(b) $(\overrightarrow{A,E}) + ?(\overrightarrow{D,E}) \doteq (\overrightarrow{A,F})$.

(c) $\frac{2}{3}(\overrightarrow{D,E}) + ?(\overrightarrow{A,C}) \doteq (\overrightarrow{D,C})$.

(d) $?(\overrightarrow{A,F}) + (\overrightarrow{A,C}) \doteq (\overrightarrow{A,A})$.

(e) $?(\overrightarrow{A,B}) + ?(\overrightarrow{E,D}) \doteq \frac{1}{3}(\overrightarrow{A,C})$.

10-3. Vectors.

Motivated by our discussion at the end of the preceding section, we begin our work in vectors with the following definitions.

DEFINITION. Any real number is called a scalar.

DEFINITION. A vector is an ordered pair of real numbers, called the components of the vector.

Notation. A vector will often be denoted by a single lower case letter with a half arrow above it, thus: $\vec{u}$. If a and b are the components of a vector we may also denote the vector by the symbol $[a,b]$. If the components of a vector are the same as the components of a directed segment $(\overrightharpoon{P,Q})$ we may denote the vector by the symbol $\overrightharpoon{PQ}$.

We should note that square brackets, rather than parentheses are used in the symbol $[a,b]$, for the vector whose components are a and b . This is done to avoid confusion with ordered pairs of real numbers such as (x,y) which are the coordinates of a point. We should also be careful not to confuse the symbol, $\overrightharpoon{PQ}$, for the vector determined by the point P and Q with the symbol, $\overrightarrow{PQ}$, for the ray determined by P and Q . The former has only a half arrow above the letters; the latter has a full arrow.

DEFINITION. If $\vec{u} = [a,b]$, the number $\sqrt{a^2 + b^2}$ is called the magnitude, or length, of $\vec{u}$.

Notation. The magnitude, or length, of $\vec{u}$ is denoted by the symbol $|\vec{u}|$.

DEFINITION. The ordered pair $[0,0]$ is called the *zero vector*.

DEFINITION. If $\vec{u} = [a,b]$ and if k is any real number, the vector

$$[ka, kb] = k[a,b] = k\vec{u}$$

is called the *product of the vector* $\vec{u}$ *and the scalar* k .

DEFINITION. Two vectors are *equal* if and only if their respective components are equal.

DEFINITION. Non-zero vectors whose components are proportional are said to be *parallel*.

The next theorem follows immediately from the definition of parallel vectors and the concept of slope.

THEOREM 10-8. If $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$ are parallel vectors, then $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ are parallel.

The following simple but very important theorem is an immediate consequence of the definition of parallel vectors, the definition of the product of a vector and a scalar, and the definition of the magnitude of a vector.

THEOREM 10-9. If $\vec{u}$ and $\vec{v}$ are parallel vectors, then

$$\vec{v} = k\vec{u}$$

where

$$k = \frac{|\vec{v}|}{|\vec{u}|} .$$

DEFINITION. If $\vec{u} = [a,b]$ and $\vec{v} = [c,d]$, the vector

$$[a + c,\ b + d] = [a,b] + [c,d] = \vec{u} + \vec{v}$$

is called the *sum* of $\vec{u}$ and $\vec{v}$.

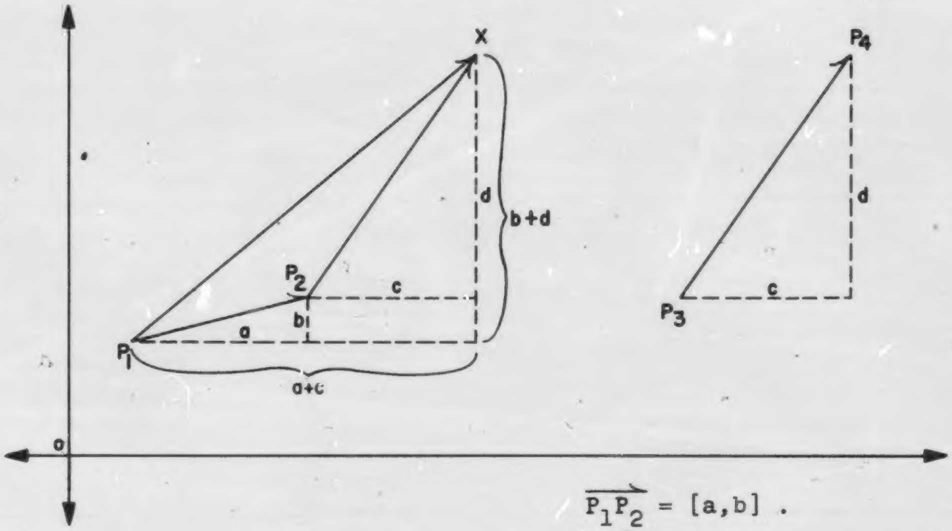
DEFINITION. The vector $\vec{u} + (-1)\vec{v}$ is called the *difference between* $\vec{u}$ *and* $\vec{v}$.

Notation. The *difference between* $\vec{u}$ *and* $\vec{v}$ is written $\vec{u} - \vec{v}$.

The following important theorem is an immediate consequence of Theorems 10-2 and 10-5.

THEOREM 10-10. The sum of the vector $\overrightarrow{P_1P_2}$ and the vector $\overrightarrow{P_3P_4}$ is the vector $\overrightarrow{P_1X}$ where X is the unique point such that $\overrightarrow{P_2X} = \overrightarrow{P_3P_4}$.

The geometrical significance of this theorem is illustrated in the following figure.



$\overrightarrow{P_1P_2} = [a,b]$.

$\overrightarrow{P_3P_4} = [c,d]$.

$\overrightarrow{P_2X} = \overrightarrow{P_3P_4}$.

$\overrightarrow{P_1X} = \overrightarrow{P_1P_2} + \overrightarrow{P_3P_4}$

$= [a + c, b + d]$.

Since vectors are not numbers, there is no reason to believe that they obey the same laws that govern the operations of arithmetic. Actually the addition of vectors and the multiplication of vectors by scalars do obey the familiar laws of arithmetic. For this reason we shall merely list these properties for reference, and verify in only two cases that they do, indeed, hold for vectors.

Properties of Vectors

1. If $\vec{u}$, $\vec{v}$ are vectors, then $\vec{u} + \vec{v}$ is a vector.
2. If $\vec{u}$, $\vec{v}$, $\vec{w}$, are any three vectors then
$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) .$$
3. There exists a vector, $\vec{0}$, such that for any vector $\vec{u}$
$$\vec{u} + \vec{0} = \vec{u} .$$
4. For every vector $\vec{u}$ there exists a vector $-\vec{u}$ such that
$$\vec{u} + (-\vec{u}) = \vec{0} .$$
5. If $\vec{u}$, $\vec{v}$ are any two vectors, then $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
6. If $\vec{u}$, $\vec{v}$ are any two vectors and k is any scalar, then
$$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v} .$$
7. If $\vec{u}$ is any vector, then $k\vec{u} = \vec{u}$ when $k = 1$.
8. If $\vec{u}$ is any vector and h , k are any two scalars, then
$$(h + k)\vec{u} = h\vec{u} + k\vec{u} .$$
9. If $\vec{u}$ is any vector and h , k are any scalars, then
$$h(k\vec{u}) = hk\vec{u} = k(h\vec{u}) .$$
10. If $\vec{u}$ is any vector and k is any scalar, then
$$|k\vec{u}| = |k| \cdot |\vec{u}| .$$

Property 1 is an immediate consequence of the definition of the sum of two vectors.

To prove Property 2, let $\vec{u} = [a,b]$, $\vec{v} = [c,d]$, and $\vec{w} = [e,f]$. Then

$$\begin{aligned}(\vec{u} + \vec{v}) + \vec{w} &= ([a,b] + [c,d]) + [e,f] \\ &= [a + c , b + d] + [e,f] \\ &= [(a + c) + e , (b + d) + f] \\ &= [a + (c + e) , b + (d + f)] \\ &= [a,b] + [c + e , d + f] \\ &= [a,b] + ([c,d] + [e,f]) \\ &= \vec{u} + (\vec{v} + \vec{w}) \text{ , as asserted.}\end{aligned}$$

Properties 3 and 4 follow immediately from the definition of the sum of two vectors and the definitions of the zero vector and the difference between two vectors.

To prove Property 5, let $\vec{u} = [a,b]$ and $\vec{v} = [c,d]$. Then

$$\begin{aligned} \vec{u} + \vec{v} &= [a,b] + [c,d] \\ &= [a + c\ ,\ b + d] \\ &= [c + a\ ,\ d + b] \\ &= [c,d] + [a,b] \\ &= \vec{v} + \vec{u}\ , \text{ as asserted.} \end{aligned}$$

The proofs of Properties 6 to 10 are very much like the two proofs we have given, and to save time we omit them. In each case, it is the corresponding property of the real numbers which appear as components that plays the decisive role in the proof.

Problem Set 10-3

1. If A, B, C are respectively (1,2), (4,3), (6,1) express each of the following vectors in component form.

   (a) $\overrightarrow{AB}$.
   (b) $\overrightarrow{BA}$.
   (c) $\overrightarrow{AA}$.
   (d) $\overrightarrow{AC}$.
   (e) $\overrightarrow{CB}$.
   (f) $\overrightarrow{CA}$.
   (g) $\overrightarrow{BC}$.

2. Same as Problem 1 if A, B, C are respectively (-1,2), (4,-3), and (-6,-1).

3. If A, B, C are respectively (1,2), (4,3), (6,1) find X so that

   (a) $\overrightarrow{AB} = \overrightarrow{CX}$.
   (b) $\overrightarrow{AX} = \overrightarrow{CB}$.
   (c) $\overrightarrow{XA} = \overrightarrow{CB}$.
   (d) $\overrightarrow{XA} = \overrightarrow{BC}$.

4. Same as Problem 3, if A, B, C are respectively (-1,2), (4,-3), (-6,-1).

5. Given $\vec{a} = [3,2]$, $\vec{b} = [-4,3]$, $\vec{c} = [5,-6]$. Determine the following.

   (a) $\vec{a} + \vec{b}$.
   (b) $\vec{a} - \vec{c}$.
   (c) $\vec{b} - \vec{a}$.
   (d) $\vec{a} + \vec{b} - \vec{c}$.
   (e) $\vec{a} - \vec{b} - \vec{c}$.
   (f) $\vec{b} + \vec{c} - \vec{a}$.
   (g) $-\vec{a} - \vec{b} - \vec{c}$.

6. Using the vectors in Problem 5, determine

   (a) $2\vec{a} + 2\vec{b}$.
   (b) $2(\vec{a} + \vec{b})$.
   (c) $-3\vec{a}$.
   (d) $2\vec{b} - \vec{c}$.
   (e) $\vec{b} - 2\vec{c}$.
   (f) $-\vec{a} + 3\vec{b} + \frac{1}{2}\vec{c}$.
   (g) $\frac{3}{4}\vec{a} - \frac{1}{4}\vec{b} + \frac{1}{4}\vec{c}$.

7. Using the vectors in Problem 5, find the real number which expresses each of the following.

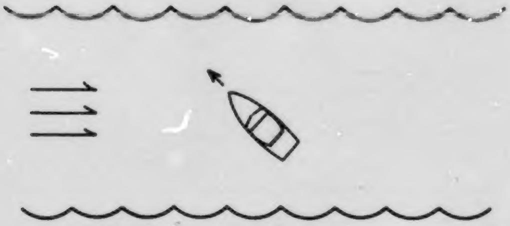
   (a) $|\vec{a}|$.
   (b) $|\vec{b} + \vec{c}|$.
   (c) $|\vec{b}|$.
   (d) $|\vec{c}|$.
   (e) $|\vec{a} - \vec{c}|$.
   (f) $|\vec{a} + \vec{b}|$.
   (g) $|\vec{a} + \vec{b} + \vec{c}|$.

8. Determine a and b so that

   (a) $[a,b] + [3,4] = [3,1]$.
   (b) $[a,b] + [2,1] = [1,-3]$.
   (c) $[1,0] = [2,4] + [a,b]$.
   (d) $[0,1] = [-3,1] + [a,b]$.
   (e) $[a,b] + [3,1] = [3,1]$.

9. Physicists have found that forces and velocities obey the law of vector addition. Physicists call this sum the resultant. Using this knowledge and a scale of 1 inch to represent 2 miles per hour, solve the following problem graphically.

A river has a 3 mile per hour current. A motor boat moves directly across the river at 5 miles per hour. How fast and in what direction would the boat be traveling if there were no current and the same power and heading were used in crossing the river?



10-4. The Two Fundamental Theorems.

Many of the applications of vectors depend upon one or the other of two theorems, which we shall now prove.

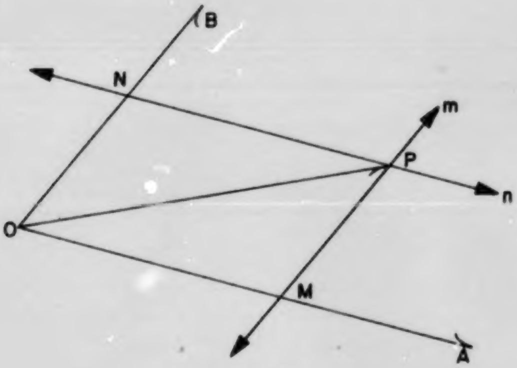
You will note in the proofs of these theorems that we refer to diagrams of geometric figures when we speak of vectors. While a vector is an ordered pair of numbers and not a set of points, the fact that a directed segment determines a vector and that a vector together with an initial point determines a directed segment, enable us to think of a directed segment as a vector.

THEOREM 10-11. If $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are two non-zero vectors which are not parallel and if $\overrightarrow{OP}$ is any vector in the plane OAB , then there exist scalars h and k such that

$$\overrightarrow{OP} = h\overrightarrow{OA} + k\overrightarrow{OB} .$$

If $\overrightarrow{OP}$ is the zero vector, it is obvious that $h = k = 0$. If $\overrightarrow{OP}$ is parallel to either $\overrightarrow{OA}$ or $\overrightarrow{OB}$, the assertion of the theorem follows immediately from Theorem 10-9. If $\overrightarrow{OP}$ is neither the zero vector nor a vector parallel to $\overrightarrow{OA}$ or

$\overrightarrow{OB}$, let m be the line which contains P and is parallel to $\overleftrightarrow{OB}$ and let n be the line which contains P and is parallel to $\overleftrightarrow{OA}$.



Let M be the intersection of m and $\overleftrightarrow{OA}$ and let N be the intersection of n and $\overleftrightarrow{OB}$. Then by Theorem 10-9,

$$\overrightarrow{OM} = h\overrightarrow{OA} \quad \text{and} \quad \overrightarrow{ON} = k\overrightarrow{OB} .$$

Finally, since $\overrightarrow{ON} = \overrightarrow{MP}$, it follows by Theorem 10-10 that

$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{ON} = h\overrightarrow{OA} = k\overrightarrow{OB} ,$$

as asserted.

Theorem 10-11 has an interesting algebraic interpretation. If $\overrightarrow{OA} = [a_1,a_2]$, $\overrightarrow{OB} = [b_1,b_2]$ and $\overrightarrow{OP} = [p_1,p_2]$, then the assertion $\overrightarrow{OP} = h\overrightarrow{OA} + k\overrightarrow{OB}$ is true if and only if there exist numbers h and k such that

$$\begin{aligned}[p_1,p_2] &= h[a_1,a_2] + k[b_1,b_2] \\ &= [ha_1,ha_2] + [kb_1,kb_2] \\ &= [ha_1 + kb_1 , ha_2 + kb_2] .\end{aligned}$$

This in turn requires that

$$ha_1 + kb_1 = p_1$$

and

$$ha_2 + kb_2 = p_2 .$$

Now we know that these equations have a unique solution for h and k unless their coefficients in one of the equations are zero or unless their coefficients in the two equations are proportional. If the coefficients are proportional, then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \text{ or equivalently } \frac{a_1}{b_1} = \frac{a_2}{b_2} .$$

But this is precisely the condition that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ should be parallel, a situation ruled out by the hypothesis of the theorem. Thus, if $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are non-zero, non-parallel vectors, whose components are known, it is possible to express a third vector, $\overrightarrow{OP}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in a purely algebraic way.

Example

Express $\vec{w} = [5,2]$ in terms of $\vec{u} = [2,3]$ and $\vec{v} = [-1,4]$.

To do this, we must determine h and k so that

$$\begin{aligned}[5,2] &= h[2,3] + k[-1,4] \\ &= [2h,3h] + [-k,4k] \\ &= [2h - k , 3h + 4k] .\end{aligned}$$

This requires that

$$2h - k = 5 \text{ and } 3h + 4k = 2 .$$

Solving these two equations simultaneously, we find

$$h = 2 , k = -1 .$$

Hence

$$\vec{w} = 2\vec{u} - \vec{v} .$$

The second of our fundamental theorems is the following.

THEOREM 10-12. If $\vec{u}$ and $\vec{v}$ are non-zero, non-parallel vectors, and if x, y, z, w are scalars such that

$$x\vec{u} + y\vec{v} = z\vec{u} + w\vec{v} ,$$

then

$$x = z \text{ and } y = w .$$

To prove this, we observe that by adding the vector $-(z\vec{u} + y\vec{v})$ to both sides of the given equation,

$$x\vec{u} + y\vec{v} = z\vec{u} + w\vec{v}$$

we obtain

$$x\vec{u} - z\vec{u} = w\vec{v} - y\vec{v}$$

or, using Property 9

$$(x - z)\vec{u} = (w - y)\vec{v} .$$

If $x - z \neq 0$ we can write

$$\vec{u} = \frac{w - y}{x - z}\vec{v} .$$

From this we conclude either that $\vec{u}$ is the zero vector (if $w - y = 0$) or else that $\vec{u}$ and $\vec{v}$ are parallel (since one is a scalar multiple of the other.) However each of these alternatives contradicts the hypothesis of the theorem. Hence $x - z$ cannot be different from zero and so $x = z$. But if $x = z$, then it follows that

$$\vec{0} = (w - y)\vec{v}$$

and since $\vec{v}$ is not the zero vector, by hypothesis, it follows that $w = y$. Hence

$$x = z \text{ and } w = y \text{ , as asserted.}$$

Problem Set 10-4

Determine x and y so that each of the following statements is true.

1. $[-6,-1] = x[3,4] + y[4,3]$.
2. $x[3,-1] + y[3,1] = [5,6]$.
3. $x[3,2] + y[2,3] = [1,2]$.
4. $x[3,2] + y[-2,3] = [5,6]$.
5. $x[3,2] + y[6,4] = [-3,-2]$.

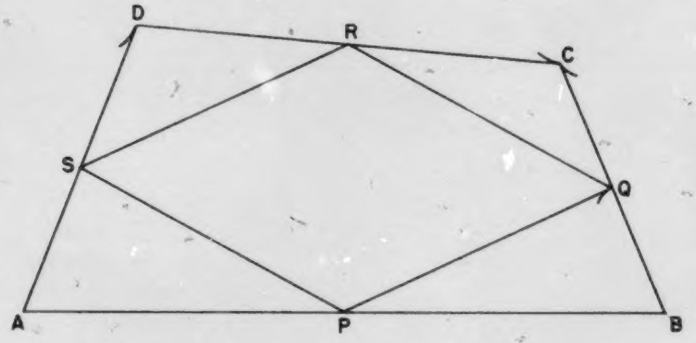
21C

10-5. Geometrical Application of Vectors.

Many theorems in geometry can be proved by means of vectors. In this section we shall present several typical examples of vector proofs of geometrical theorems.

THEOREM 10-13. The midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

Proof: Let A, B, C, D be the vertices of the quadrilateral and let P, Q, R, S be the midpoints of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, respectively.



By hypothesis, $\overrightarrow{SD} = \frac{1}{2}\overrightarrow{AD}$, $\overrightarrow{DR} = \frac{1}{2}\overrightarrow{DC}$, $\overrightarrow{PB} = \frac{1}{2}\overrightarrow{AB}$, $\overrightarrow{BQ} = \frac{1}{2}\overrightarrow{BC}$.

Hence $\overrightarrow{SR} = \overrightarrow{SD} + \overrightarrow{DR} = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{DC})$

and $\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC})$.

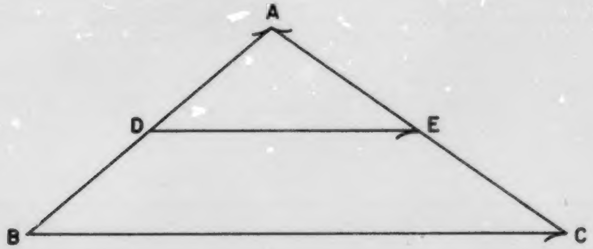
Moreover $\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AB} + \overrightarrow{BC}$, since each is equal to $\overrightarrow{AC}$.

Therefore $\overrightarrow{SR} = \overrightarrow{PQ}$,

which implies that $SR = PQ$ and $\overleftrightarrow{SR} \parallel \overleftrightarrow{PQ}$. Hence PQRS is a parallelogram, as asserted.

THEOREM 10-14. The segment joining the midpoints of two sides of a triangle is parallel to the third side and the length of the segment is one half the length of the third side.

Proof: Let A, B, C be the vertices of the triangle and let D and E be the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively.



By hypothesis $\overrightarrow{DA} = \frac{1}{2}\overrightarrow{BA}$ and $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC}$.

$$\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC})$$

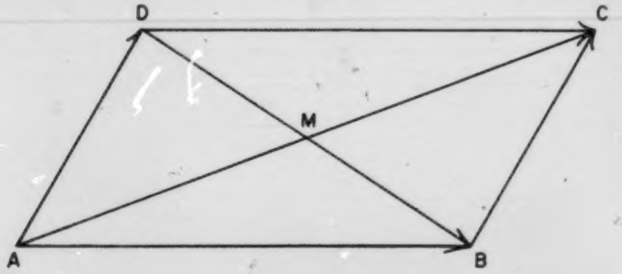
Hence

$$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$$

which implies that $DE = \frac{1}{2}BC$ and $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$ as asserted.

THEOREM 10-15. A quadrilateral is a parallelogram if and only if its diagonals bisect each other.

Proof: Let ABCD be the parallelogram and let M be the intersection of its diagonals.



Let $\overrightarrow{AD} = \vec{u}$ and $\overrightarrow{DC} = \vec{v}$. Then $\overrightarrow{AC} = \vec{u} + \vec{v}$ and $\overrightarrow{DB} = \vec{v} - \vec{u}$. Since A, M, C are collinear, $\overrightarrow{AM}$ is some scalar multiple of $\overrightarrow{AC}$, say $x(\vec{u} + \vec{v})$. Similarly $\overrightarrow{DM} = y(\vec{v} - \vec{u})$. Since $\overrightarrow{AD} + \overrightarrow{DM} = \overrightarrow{AM}$, we have

$$\vec{u} + \vec{y}(\vec{v} - \vec{u}) = x(\vec{u} + \vec{v}) ,$$

or, collecting like terms,

$$(1 - y - x)\vec{u} + (y - x)\vec{v} = \vec{0} .$$

Therefore $1 - x - y = 0$ and $y - x = 0$.

Solving these simultaneously we find

$$x = y = \frac{1}{2} .$$

Hence

$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC} \quad \text{and} \quad \overrightarrow{DM} = \frac{1}{2}\overrightarrow{DB} .$$

These imply $AM = \frac{1}{2}AC$ and $DM = \frac{1}{2}DB$, as asserted.

Now let ABCD be any quadrilateral with its diagonals bisecting each other at M so that DM = MB and AM = MC . Let $\vec{t} = \overrightarrow{AM} = \overrightarrow{MC}$ and $\vec{w} = \overrightarrow{DM} = \overrightarrow{MB}$. Then $\overrightarrow{AB} = \vec{t} + \vec{w}$ and $\overrightarrow{DC} = \vec{w} + \vec{t}$; therefore $\overrightarrow{AB} = \overrightarrow{DC}$, which implies that AB = DC and $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$. Hence ABCD is a parallelogram.

Problem Set 10-5

1. The segment joining the midpoints of the non-parallel sides of a trapezoid is called the median of the trapezoid. Prove that the median of a trapezoid is parallel to the bases and has a length equal to one-half the sum of the lengths of the bases.
2. Let ABCD be a trapezoid, with $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, and E, F the midpoints of $\overline{AC}$, $\overline{BD}$, respectively. Prove that $EF = \frac{1}{2}|AB - DC|$.
3. Prove that the medians of a triangle are concurrent at the point which trisects each median.

4. Let ABCD be a parallelogram, with E the midpoint of $\overline{AB}$, and $\overline{DE}$ intersecting $\overline{AC}$ at F. Prove that F is a point of trisection of $\overline{AC}$.

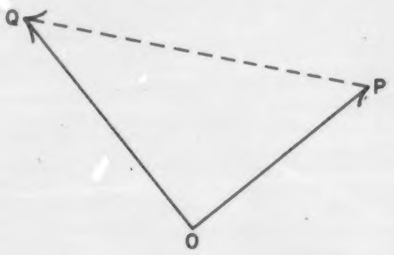
5. Let ABCD be a parallelogram, with E the point on $\overline{AB}$ such that $AE = \frac{1}{m}AB$, with $\overline{DE}$ intersecting $\overline{AC}$ at F. Prove that $AF = \frac{1}{m+1}AC$.

10-6. The Scalar Product of Two Vectors.

In Section 10-3 we defined what we meant by two parallel vectors. It is now convenient to introduce the ideas of perpendicular vectors.

> DEFINITION. Two vectors, $\overrightarrow{P_1Q_1}$ and $\overrightarrow{P_2Q_2}$ are said to be perpendicular if $\overleftrightarrow{P_1Q_1}$ is perpendicular to $\overleftrightarrow{P_2Q_2}$.

In many applications it is important to be able to tell whether or not two vectors are perpendicular. To develop a procedure for deciding this question, consider two non-zero vectors $\overrightarrow{OP} = [p_1, p_2]$ and $\overrightarrow{OQ} = [q_1, q_2]$. These will be perpendicular if and only if $\triangle POQ$ has a right angle at O. By the Pythagorean Theorem, this will be the case



if and only if

$$|\overrightarrow{PQ}|^2 = |\overrightarrow{OP}|^2 + |\overrightarrow{OQ}|^2 .$$

Now $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$.

Hence $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$,

and therefore $\overrightarrow{PQ} = [q_1 - p_1 , q_2 - p_2]$.

Now, recalling the definition of the magnitude or length of a vector, we can write $|\overrightarrow{PQ}|^2 = |\overrightarrow{OP}|^2 + |\overrightarrow{OQ}|^2$ in the form

$$(q_1 - p_1)^2 + (q_2 - p_2)^2 = (p_1{}^2 + p_2{}^2) + (q_1{}^2 + q_2{}^2) ;$$

or, expanding and collecting terms,

$$q_1{}^2 - 2q_1p_1 + p_1{}^2 + q_2{}^2 - 2q_2p_2 + p_2{}^2 = p_1{}^2 + p_2{}^2 + q_1{}^2 + q_2{}^2 .$$

Hence,

$$-2(p_1q_1 + p_2q_2) = 0$$

or

$$p_1q_1 + p_2q_2 = 0 .$$

Thus, since the preceding steps are all reversible, we have established the following important theorem.

THEOREM 10-16. Two non-zero vectors are perpendicular if and only if the sum of the products of their respective components is zero.

The number $p_1q_1 + p_2q_2$ obtained from the components of the vectors $[p_1,p_2]$ and $[q_1,q_2]$ is a very important quantity, and it is convenient to have a name for it.

DEFINITION. If $\vec{u} = [p_1,p_2]$ and $\vec{v} = [q_1,q_2]$, the number $p_1q_1 + p_2q_2$ is called the scalar product of $\vec{u}$ and $\vec{v}$.

Notation. The scalar product of $\vec{u}$ and $\vec{v}$ is denoted by the symbol $\vec{u} \cdot \vec{v}$ (read "u dot v") .

We should understand that the scalar product of two vectors is a scalar and not a vector. The name scalar product is used to emphasize this fact.

There are several important algebraic properties of the scalar product of two vectors with which we should be familiar. These are not hard to prove, and we leave the proofs of the first two as exercises.

Property 1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

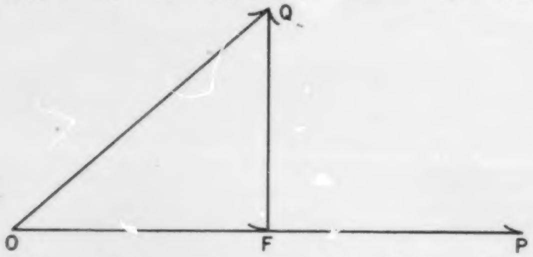
Property 2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

Property 3. If k is a scalar, $\vec{u} \cdot (k\vec{w}) = (k\vec{u}) \cdot \vec{w} = k(\vec{u} \cdot \vec{w})$.

Property 4. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

We have already seen (Theorem 10-16) that two non-zero vectors are perpendicular if and only if their scalar product is zero. However, whether two vectors are perpendicular or not, their scalar product has an interesting geometrical interpretation. To discover this let $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ be two non-zero vectors and let $\overrightarrow{OF}$ be a scalar multiple of $\overrightarrow{OP}$, say $\overrightarrow{OF} = k\overrightarrow{OP}$. Then

$$\overrightarrow{OF} + \overrightarrow{FQ} = \overrightarrow{OQ} \quad \text{or} \quad \overrightarrow{FQ} = \overrightarrow{OQ} - \overrightarrow{OF} = \overrightarrow{OQ} - k\overrightarrow{OP} .$$



We note that $\overrightarrow{FQ} = \vec{0}$ if and only if $\overrightarrow{OQ} = k\overrightarrow{OP}$, which means that $\overrightarrow{OQ}$ and $\overrightarrow{OP}$ are parallel. Now if $\overrightarrow{FQ} \neq \vec{0}$, $\overrightarrow{FQ}$ and $\overrightarrow{OP}$ will be perpendicular if and only if

$$\overrightarrow{OP} \cdot \overrightarrow{FQ} = \overrightarrow{OP} \cdot (\overrightarrow{OQ} - k\overrightarrow{OP}) = 0 ,$$

or, using Properties 2 and 3 for scalar products,

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} - k\overrightarrow{OP} \cdot \overrightarrow{OP} = 0 .$$

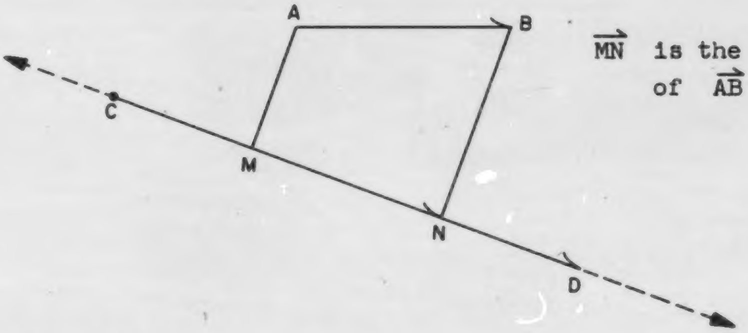
Let k' be the value of k determined by this equation. Then

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} = k'\overrightarrow{OP} \cdot \overrightarrow{OP} = k'|\overrightarrow{OP}|^2 = \begin{cases} |k'\overrightarrow{OP}|\ |\overrightarrow{OP}| & \text{if } k' > 0 \\ 0 & \text{if } k' = 0 \\ -|k'\overrightarrow{OP}|\ |\overrightarrow{OP}| & \text{if } k' < 0 . \end{cases}$$

To interpret this result it is convenient to introduce the following definition.

> DEFINITION. By the projection of a vector $\overrightarrow{AB}$ on a vector $\overrightarrow{CD}$ we mean the vector $\overrightarrow{MN}$, where M and N are, respectively, the feet of the perpendiculars from A and B to the line $\overleftrightarrow{CD}$.

The following figure illustrates this definition.



$\overrightarrow{MN}$ is the projection of $\overrightarrow{AB}$ on $\overrightarrow{CD}$.

Now k' is the value assumed by k when $\overrightarrow{FQ} \perp \overrightarrow{OP}$. Hence $k'\overrightarrow{OP}$ is the projection of $\overrightarrow{OQ}$ on $\overrightarrow{OP}$ and $|k'\overrightarrow{OP}|$ is the length of this projection. Moreover, if $\overrightarrow{FQ} = \overrightarrow{0}$, then $\overrightarrow{OQ}$ is parallel to $\overrightarrow{OP}$, and $\overrightarrow{OQ}$ is its own projection on $\overrightarrow{OP}$. Hence in all cases, $k'\overrightarrow{OP}$ is the projection of $\overrightarrow{OQ}$ on $\overrightarrow{OP}$.

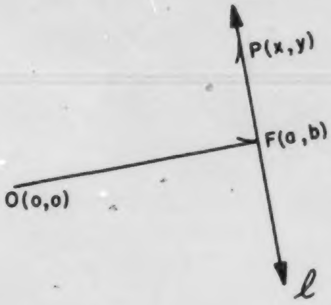
Returning now to the expression for $\overrightarrow{OP} \cdot \overrightarrow{OQ}$ which we derived above, and noting the symmetry of the scalar product guaranteed by Property 1, it follows that, except for sign, the scalar product of two vectors, $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, is equal to either:

(a) The length of $\overrightarrow{OP}$ multiplied by the length of the projection of $\overrightarrow{OQ}$ on $\overrightarrow{OP}$, or

(b) The length of $\overrightarrow{OQ}$ multiplied by the length of the projection of $\overrightarrow{OP}$ on $\overrightarrow{OQ}$.

The sign of the scalar product is positive if $k' > 0$, that is, if F lies on the ray $\overrightarrow{OP}$, and negative if $k' < 0$, that is, if F lies on the ray opposite to $\overrightarrow{OP}$.

As an example of the use of the scalar product in coordinate geometry, let O(0,0) and F(a,b) be two distinct points in the xy-plane and let ℓ be the line which is perpendicular to $\overleftrightarrow{OF}$ at F. If P(x,y) is any point of ℓ distinct from F, ℓ will be perpendicular to $\overleftrightarrow{OF}$ if and only if



$\overrightarrow{OF} \cdot \overrightarrow{FP} = 0$. Now $\overrightarrow{OF} = [a,b]$ and $\overrightarrow{FP} = [x - a, y - b]$. (Why?) Hence $\overrightarrow{OF} \cdot \overrightarrow{FP} = 0$ can be written
$a(x - a) + b(y - b) = 0$ or

$$ax + by = a^2 + b^2 .$$

By direct substitution, it is easy to verify that this equation is also satisfied by the coordinates of F. Hence this equation is an equation of the line ℓ. By an almost identical argument it can be shown that if ℓ contains O, an equation of ℓ is

$$ax + by = 0 .$$

It is interesting to compare this discussion with the derivation of an equation of a plane in Section 9-9.

Problem Set 10-6

In each of the following problems determine the scalar product and from it tell whether the two vectors are perpendicular.

1. [-5,2] , [6,15] .
2. [6,3] , [-3,-2] .
3. [-5,-2] , [3,5] .
4. [-2,3] , [6,-4] .
5. [3,-2] , [-3,2] .
6. [7,3] , [3,-7] .
7. [2,-4] , [4,6] .
8. [12,2] , [-4,-24] .
9. [6,-3] , [2,1] .
10. [9,2] , [-2,9] .
11. Using the scalar product, show that the line through P(3,5) and Q(7,-1) is perpendicular to the line through R(0,0) and S(12,8) .
12. Using the scalar product, show that P(5,7) , Q(8,-5) , and R(0,-7) are the vertices of a right triangle.
13. By using Properties 1 and 2, show that $(\vec{u} - \vec{v}) \cdot (\vec{w} - \vec{z}) = \vec{u} \cdot \vec{w} - \vec{u} \cdot \vec{z} - \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{z}$.
14. Show that an equation of a line through the origin is $ax + by = 0$.
15. Prove Properties 1 and 2 of the algebraic properties of scalar products.

10-7. Summary.

A directed segment is the mathematical entity which corresponds to a displacement in the physical world. It differs from a segment in that one of its endpoints is identified as an origin and the other as a terminus. A directed segment therefore tells both a length and a direction. After defining equivalent directed segments we introduced a vector as a set of equivalent directed segments. Since equivalent directed segments have the same components we can consider a vector to be an ordered pair of numbers, and this is how we defined a vector. We used vectors to prove some geometric theorems. These proofs were sometimes simple due to the fact that the algebra of multiplying vectors by scalars is similar to the algebra we studied in previous grades. The chapter ended with scalar multiplication which enables us to prove two lines perpendicular and to find the projection of one vector on another.

Review Problems

1. Given ABCD is a parallelogram and E and F trisection points of $\overline{AC}$, such that E is between A and F. Prove that DEBF is a parallelogram.

2. Given parallelogram ABCD, and E and F so chosen that AB + BE = AE and CD + DF = CF and BE = FD. Show that AECF is a parallelogram.

3. Show that the points $P(6,8)$, $Q(0,-2)$, $R(-3,-7)$ are collinear.

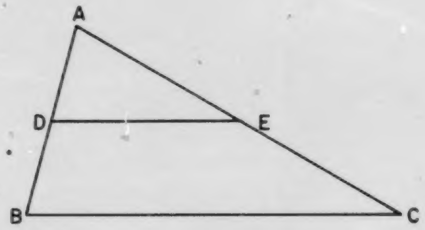
4. Show that $P(4,0)$, $Q(7,8)$, $R(0,10)$ and $S(-3,2)$ are the vertices of a parallelogram.

5. If $\vec{a} = [4,0]$, $\vec{b} = [-3,2]$, $\vec{c} = [7,8]$ find

   (a) $\vec{a} + \vec{b}$. (c) $\vec{a} + \vec{b} + \vec{c}$.

   (b) $\vec{a} - \vec{c}$. (d) $\vec{b} - \vec{c}$.

6. In the figure D and E are midpoints of $\overline{AB}$ and $\overline{AC}$ respectively.



   (a) $(\overrightarrow{A,D}) = ?(\overrightarrow{A,B})$.

   (b) $?(\overrightarrow{A,D}) + (\overrightarrow{B,C}) = ?(\overrightarrow{A,E})$.

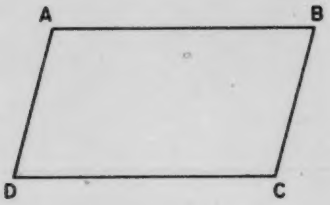
   (c) $(\overrightarrow{A,D}) + (\overrightarrow{D,E}) = ?(\overrightarrow{A,C})$.

   (d) $(\overrightarrow{B,C}) + (\overrightarrow{C,A}) = ?$

   (e) $(\overrightarrow{D,B}) + (\overrightarrow{B,C}) = (\overrightarrow{D,E}) + ?(\overrightarrow{A,C})$.

7. A, B, C, D are vertices of a parallelogram. List all the directed line segments they determine, and indicate which pairs are equivalent.

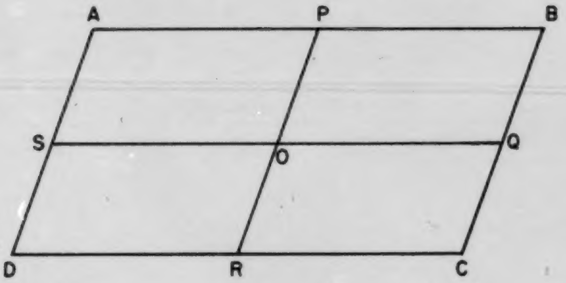
8.



If ABCD is a parallelogram, express $(\overrightarrow{D,B})$ in terms of

(a) $(\overrightarrow{D,C})$ and $(\overrightarrow{D,A})$.
(b) $(\overrightarrow{D,C})$ and $(\overrightarrow{C,B})$.
(c) $(\overrightarrow{A,B})$ and $(\overrightarrow{B,C})$.
(d) $(\overrightarrow{A,B})$ and $(\overrightarrow{A,D})$.
(e) $(\overrightarrow{B,A})$ and $(\overrightarrow{B,C})$.

9. ABCD is a parallelogram and P, Q, R, S are the midpoints of the sides.



For each of the following directed line segments, find an equivalent directed line segment of the form

$$r(\overrightarrow{O,Q}) + s(\overrightarrow{O,P}) .$$

(a) $(\overrightarrow{O,B})$.
(b) $(\overrightarrow{O,C})$.
(c) $(\overrightarrow{O,D})$.
(d) $(\overrightarrow{O,A})$.
(e) $(\overrightarrow{D,B})$.
(f) $(\overrightarrow{A,C})$.
(g) $(\overrightarrow{C,A})$.
(h) $(\overrightarrow{B,D})$.

10. Determine x and y so that

(a) $x[3,1] + y[2,-1] = [13,1]$.
(b) $x[2,3] + y[3,1] = [7,0]$.
(c) $x[3,6] + y[4,2] = [4,2]$.
(d) $x[-3,2] + y[1,1] = [0,0]$.
(e) $x[1,2] + y[-1,1] = [6,6]$.

11. If $\vec{a} = [3,1]$, $\vec{b} = [2,-1]$, find

(a) $|\vec{a}|$.
(b) $|\vec{a} + \vec{b}|$.
(c) $|\vec{b}|$.
(d) $|\vec{a} - \vec{b}|$.
(e) $|\vec{b} - \vec{a}|$.

12. If A, B, C are respectively (4,2) , (6,3) , (2,1) express the following vectors in component form.

(a) $\overrightarrow{AB}$.
(b) $\overrightarrow{BA}$.
(c) $\overrightarrow{AC}$.
(d) $\overrightarrow{CB}$.
(e) $\overrightarrow{BC}$.

13. Determine the scalar product of

(a) [4,2] , [2,-1] .
(b) [1,1] , [3,1] .
(c) [3,2] , [4,2] .
(d) [3,6] , [1,1] .
(e) [1,2] , [-1,1] .

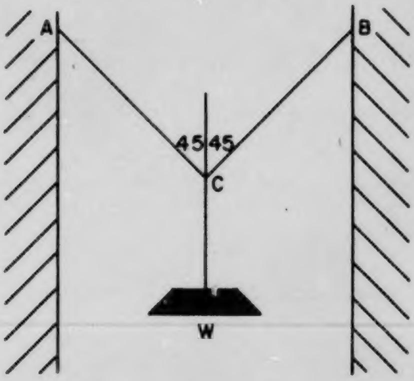
14. In a cube what is the maximum number of equivalent directed line segments?

15. In a trapezoid what is the maximum number of equivalent directed line segments?

16. A man is standing on top of a hill. He weighs 200 pounds. Represent as a vector each of the following: (Use a scale of 1 inch = 200 pounds.)

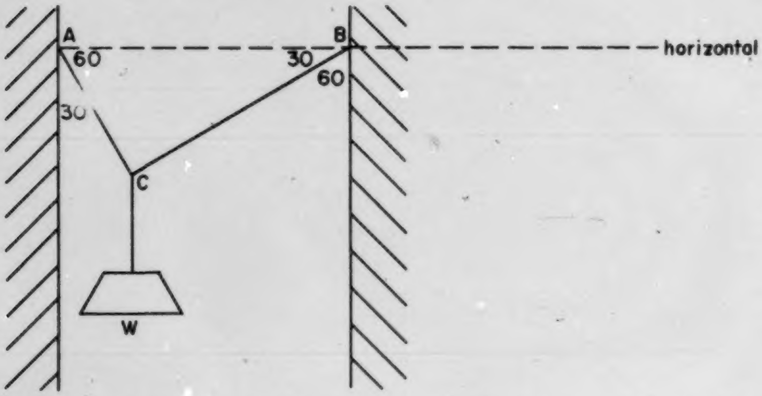
(a) The downward pull of the earth's gravity on him.
(b) The upward push of the hill on him.

17. An object is suspended by ropes as shown in the figure.



If the object weighs 10 pounds, what is the force exerted on the junction C by the rope $\overline{CB}$?

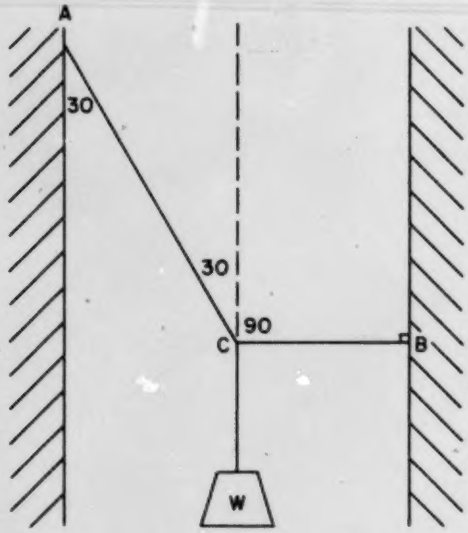
18. A weight of 1000 pounds is suspended from wires as shown in the figure.



(a) What force does the wire $\overline{AC}$ exert on the junction C ?

(b) What force does the wire $\overline{BC}$ exert on C ?

19. A 5000 pound weight is suspended as shown in the figure. Find the tension in each of the ropes $\overline{CA}$, $\overline{CB}$, and $\overline{CW}$.



20. A ship sails east at 20 miles per hour. A man walks across its deck toward the south at 4 miles per hour. What is the man's velocity relative to the water?

Chapter 11

POLYGONS AND POLYHEDRONS

11-1. *Introduction*.

In the physical world nature abounds in geometric shapes. Many of these shapes are representations of polygons and polyhedrons. For example, the sections of a honeycomb are hexagonal; each snow crystal is in the shape of a tiny hexagon; diamonds are in the form of regular octahedrons; salt crystals appear to be tiny cubes; and quartz crystals have the shape of hexagonal pyramids.

Man uses the shapes of regular polygons in designing formal landscapes, in making bolt heads, chickenwire, stop signs, and linoleum tiles. Box cartons, buildings, and skyscrapers take the form of prisms and other polyhedrons.

In this chapter, we continue our study of polygons with special emphasis on the area of polygonal-regions. It is interesting to note that one of the first practical uses of geometry was that of finding area. Many people think that geometry had its origin in the fourteenth century B. C. along the banks of the Nile River. At that time the king of Egypt divided the land into plots and obtained his revenue from the annual rent which the landholders were required to pay. Each year the Nile River overflowed and carried away portions of soil. This necessitated a remeasurement of the land so that the rent demanded of an individual that year would be proportional to the land which he held.

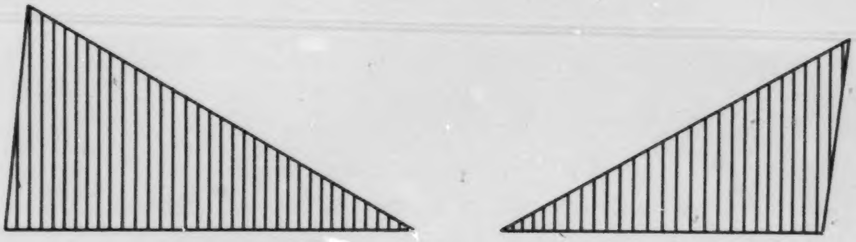
It is also interesting to note that the word *geometry* comes from two Greek words *ge* meaning "earth" and *metrein* meaning "to measure." Hence the first meaning of the word geometry was "earth measurement."

Today the study of area is also important. Land is bought and sold by the acre; the floor space of a building is considered in determining the rent of an office, factory, or storeroom; the area of the wing of an airplane is important in designing the airplane; painters, bricklayers, surveyors, map makers, and interior decorators must know how to calculate the area of simple geometric figures.

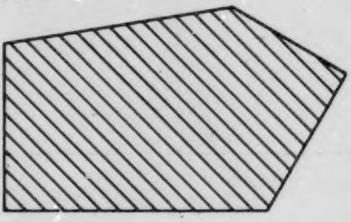
In the latter part of this chapter, we introduce figures in three dimensions which are closely analogous to the polygons we have studied in two dimensions. Each of these figures is called a polyhedron. We shall investigate some of the interesting properties of this set of surfaces. However, the study of the measure of a polyhedral-region will be deferred.

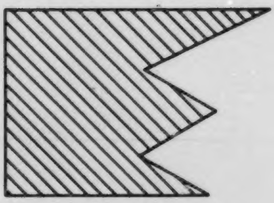
11-2. Polygonal-Regions.

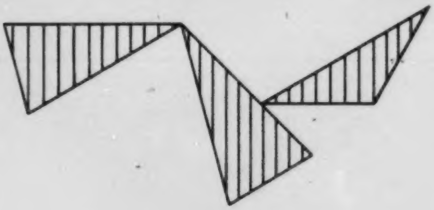
A triangular region consists of a triangle and its interior. Each of the following diagrams represents a triangular-region.

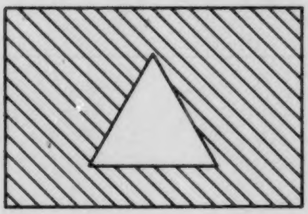


A polygonal-region is a figure in a plane, like one of these four:

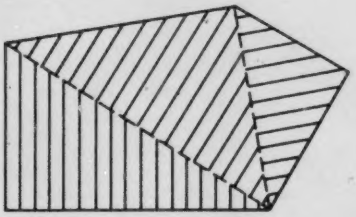


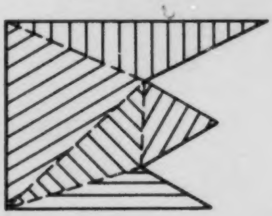






Notice in particular that a polygonal-region may have one or more "holes" in it. A polygonal-region can be "cut up" into triangular regions. For example, each of the first two polygonal-regions shown above is "cut up" in the diagrams below.

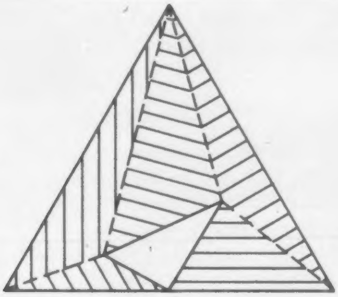


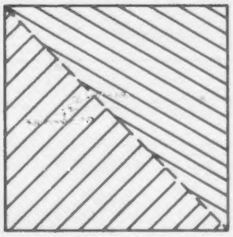


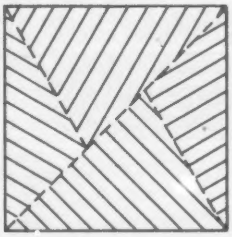
DEFINITIONS. A triangular region is the union of a triangle and its interior.

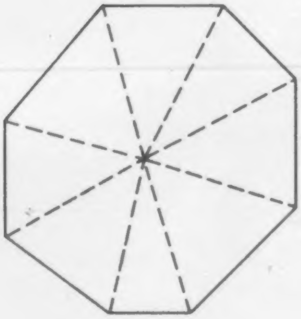
A polygonal-region is the union of a finite number of coplanar triangular regions.

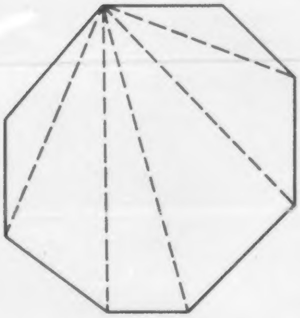
Each of the following figures pictures a polygonal-region as a union of triangular regions.







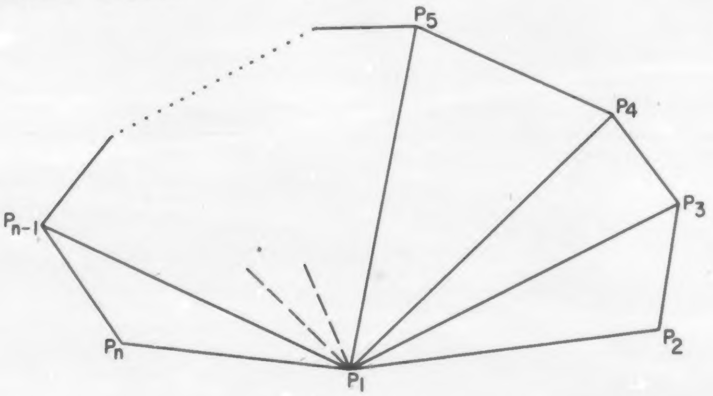




The preceding two pairs of pictures suggest that a polygonal-region can be considered as a union of triangular-regions in more than one way. Note that we often do not shade a polygonal-region in a picture, in case the context makes clear that we are considering the polygonal-region rather than the polygon which is the "boundary" of the polygonal-region.

One of the above diagrams shows five diagonals of a convex polygon with eight sides. These five diagonals have a common endpoint and they "cut up" the polygonal-region so that we see the polygonal-region as a union of six triangular-regions. Noting that $5 = 8 - 3$ and $6 = 8 - 2$, we are ready to consider the general situation.

Consider any convex polygon, say $P_1P_2 \ldots P_n$. We wish to observe that the union of the convex polygon and its interior is the union of $n - 2$ triangular-regions and is therefore a polygonal-region.



(In the figure, the dots indicate other possible vertices and sides, because we do not know what the number n is.) The number of sides of the polygon is n. Since the polygon is a convex polygon, the $n - 1$ rays $\overrightarrow{P_1P_2}$, $\overrightarrow{P_1P_3}$, ..., $\overrightarrow{P_1P_n}$, are concurrent in that order. Two of the corresponding segments, namely $\overline{P_1P_2}$ and $\overline{P_1P_n}$, are sides of the polygon, while the remaining $n - 3$ segments, $\overline{P_1P_3}$, $\overline{P_1P_4}$, ..., $\overline{P_1P_{n-1}}$, are all of the diagonals with one endpoint at P_1. (Some of these diagonals are shown in the figure, and other possible diagonals are suggested by - - - - .)

These diagonals and the sides of the polygon give us triangles, $\triangle P_1P_2P_3$, $\triangle P_1P_3P_4$, ... , $\triangle P_1P_{n-1}P_n$, the number of which is $n - 2$. The union of these triangles and their interiors is the union of $P_1P_2 \ldots P_n$ and its interior. Thus the union of a convex polygon and its interior is a polygonal-region.

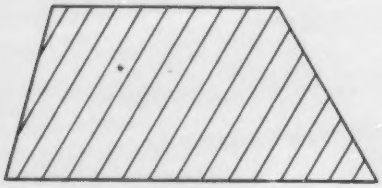
Furthermore, we observe that the $n - 2$ triangles mentioned above have the property that the interiors of no two of them intersect. Hence, if a convex polygon has n sides, the union of the polygon and its interior is the union of $n - 2$ triangles and their interiors such that the interiors of any two of the triangles do not intersect.

DEFINITIONS. If a polygonal-region is the union of a convex polygon and its interior, then the polygon is called the boundary of the polygonal-region and the interior of the polygon is called the interior of the polygonal-region.

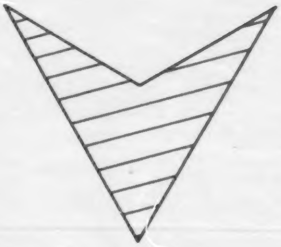
In this chapter, we make use of triangular-regions in two ways: (1) to determine the sum of the measures of the angles of a convex polygon, and (2) to study the areas of various polygonal-regions.

Problem Set 11-2

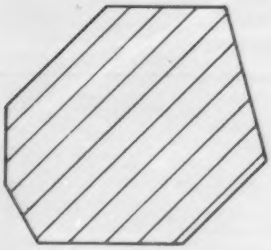
1. Show that each of the following is a polygonal-region. More specifically, show that each is a union of triangular-regions such that the interiors of any two of the triangular regions do not intersect. Try to find the smallest number of triangular-regions in each case. (Note: The boundary in Part (d) is a star-shaped polygon, and each side of the polygon is collinear with another side of the polygon. The boundary in Part (g) is a polygon having two nonconsecutive sides which are collinear.)



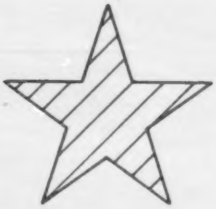
(a)



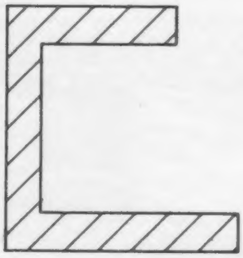
(b)



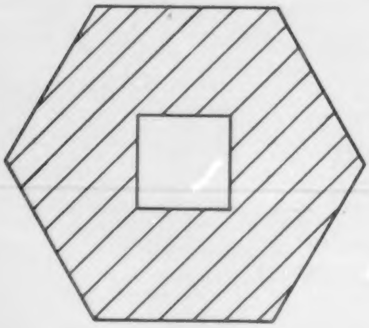
(c)



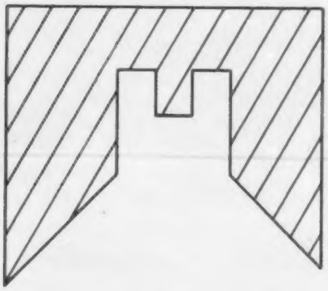
(d)



(e)

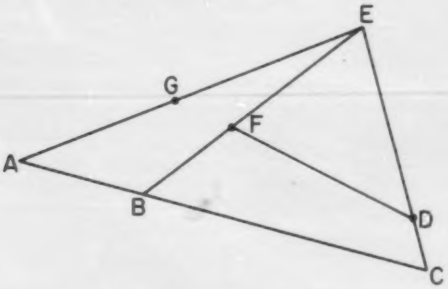


(f)



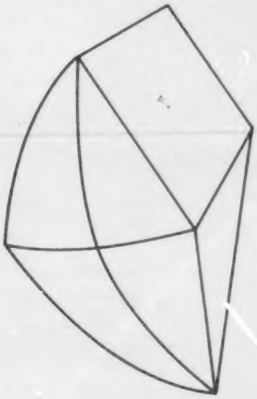
(g)

2. In the following figure, A, B, C, D, E, F, G are called vertices, the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EG}$, $\overline{GA}$, $\overline{EF}$, $\overline{FD}$, $\overline{FB}$ are called edges, and the polygonal-regions ABE, FED, BCDF are called faces. The exterior of the figure will also be considered as a face.

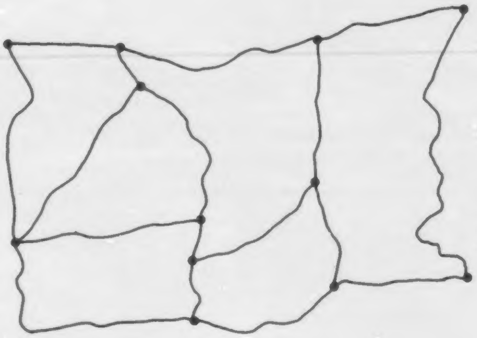


Let the number of faces be f , the number of vertices be v , and the number of edges be e . In a theorem which was originated by a famous mathematician, Euler, and which refers to figures of which the above figure is an example, there occurs the number f - e + v . Using the figure, let's compute f - e + v . You should see that f = 4 , v = 7 , e = 9 , and this gives us f - e + v = 2 . Using the two figures below, compute f - e + v . Notice that the edges are not necessarily segments.

(a)



(b) Suppose this figure to be a section of a map showing counties:



(c) What do you observe in the results of the three computations?

(d) In Part (a) take a point in the interior of the quadrilateral and draw segments from each of the four vertices to the point. How does this affect the number $f - e + v$? Can you explain why?

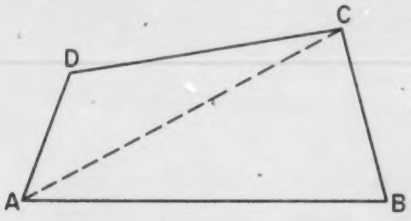
(e) Take a point in the exterior of the figure of Part (a) and connect it to the two nearest vertices. How does this affect the computation?

If you are interested in this problem and would like to pursue it further, you will find it discussed in "The Enjoyment of Mathematics" by Rademacher and Toeplitz and in "Fundamental Concepts of Geometry" by Meserve.

11-3. Sum of the Measures of the Angles of a Convex Polygon.

In Chapter 6 we proved that the sum of the measures of the angles of a triangle is 180 . As an application of this important theorem we studied the sum of the measures of the angles of a convex quadrilateral. Let us review the method which we used (see Theorem 6-13 and its proof), but let us express the ideas with the aid of the new terminology introduced in the preceding section.

If the quadrilateral ABCD is a convex quadrilateral, then polygonal-region ABCD is the union of the two triangular-regions ABC and ACD .

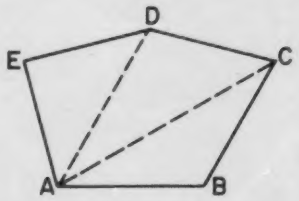


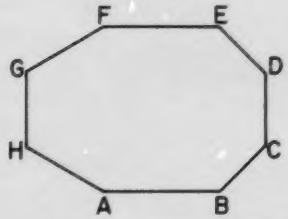
We showed in the proof of Theorem 6-13 that the sum of the measures of the four angles of the quadrilateral is the same as the measures of six angles, three from each of the two triangles. Thus we obtained the number 2 · 180 , or 360 , as the sum of the measures of the angles of the convex quadrilateral.

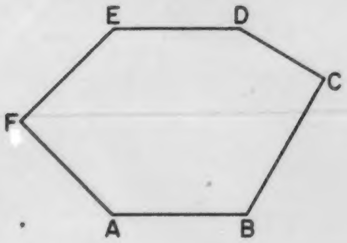
We wish to extend this discussion to the case of a convex polygon of any number of sides. The following exploratory problem utilizes our observations in the preceding section concerning the representation of a polygonal-region as the union of triangular-regions, no two of whose interiors intersect.

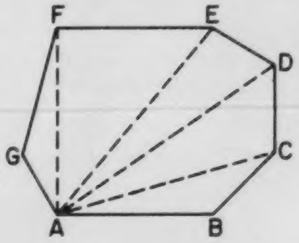
Exploratory Problem

Consider the diagonals from A in each of the convex polygons pictured below. By a procedure similar to the one we used with the quadrilateral, find the sum of the measures of the angles of each polygon. Summarize the results in a data table as shown.







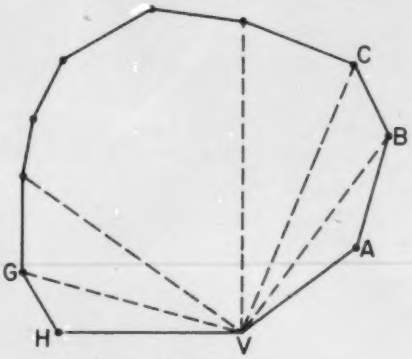


Number of sides of convex polygon	Number of diagonals from A	Number of triangular regions	Sum of measures of angles of the polygon
4	1	2	2 × 180 = 360
5	2		
6	3		
7	4		
8	5		
n			

This exploratory problem leads us to the following important result.

THEOREM 11-1. The sum of the measures of the angles of a convex polygon of n sides is $(n - 2) \cdot 180$.

Proof: Let V be any vertex of the given convex polygon with n sides, and let the polygon be $VABC...GH$. There are $n - 3$ diagonals from the vertex V . The union of the triangular regions AVB , BVC , ... , GVH is the polygonal-region $VABC...GH$. There are $n - 2$ of these triangular-regions, and the interiors of no two of them intersect. The total measure of all the angles of these triangles is $(n - 2) \cdot 180$. On the other hand, the total measure of all the angles of these triangles is the same as the sum of the measures of all the angles of polygon $VABC...GH$. (Why?)



Corollary 11-1-1. The measure of each angle of a regular polygon of n sides is

$$\frac{(n - 2)180}{n} , \text{ or } 180 - \frac{360}{n} .$$

Proof: A regular polygon of n sides has n angles, and all of these angles have the same measure. Hence each has measure $\frac{1}{n}(n - 2)(180)$. Now

$$\frac{(n - 2) \cdot 180}{n} = (1 - \frac{2}{n}) \cdot 180 = 180 - \frac{360}{n} .$$

The notion of an exterior angle of a triangle, as described in Chapter 5, may be extended in a natural manner to polygons of more than three sides.

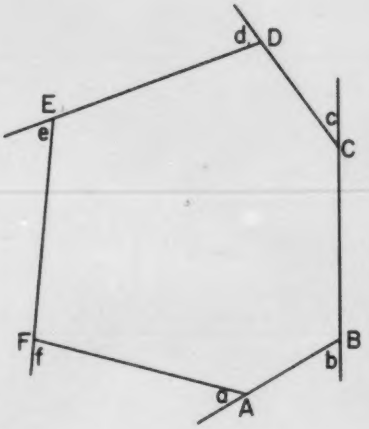
DEFINITIONS. Let V be any vertex of a convex polygon.

The angle of the polygon with vertex V is sometimes called the interior angle of the polygon at V .

Either angle which forms a linear pair with the interior angle of the polygon at V is called an exterior angle of the polygon at V .

THEOREM 11-2. For any convex polygon of n sides, the sum of the measures of exterior angles, one at each vertex of the polygon, is 360 .

Proof: At each vertex of the polygon, choose an exterior angle. The chosen exterior angle and the interior angle at that vertex are supplementary; the sum of their measures is 180 . The sum of the measures of all the interior angles and all the chosen exterior angles is $n \cdot 180$. The sum of the measures of the interior angles is $(n - 2)180$. By subtraction, the sum of the measures of the selected exterior angles is



$$n \cdot 180 - (n - 2) \cdot 180 = [n - (n - 2)]180 = 2 \cdot 180 = 360 .$$

Corollary 11-2-1. The measure of each exterior angle of a regular polygon of n sides is $\frac{360}{n}$.

Proof: This statement is an immediate consequence either of Theorem 11-2 or of Corollary 11-1-1. Why?

Problem Set 11-3

1. Find the sum of the measures of the interior angles and the sum of the measures of the exterior angles of a polygon, one exterior angle at each vertex, if the number of sides of the polygon is:

   (a) 12 (b) 22

2. The sum of the measures of the interior angles of a certain regular polygon is 1080 . By Theorem 11-1,

   $$(n - 2)180 = 1080 .$$

   Hence

   $$n - 2 = \frac{1080}{180} = 6$$

   and

   $$n = 8 .$$

   Thus the number of sides of the polygon is 8 .

   Find the number of sides of a regular polygon if the sum of the measures of the interior angles is:

   (a) 540 (b) 900 (c) 2700

3. Consider a regular nonagon (nine sides). The measure of each exterior angle is $\frac{360}{9}$ or 40 . The interior angle and an exterior angle at each vertex are a linear pair of angles.

   (a) What is the measure of each interior angle of this polygon?

   (b) What is the sum of the measures of all the interior angles?

4. Use two methods to find the measure of each interior angle of a regular polygon of 12 sides.

5. We know that an interior angle and an exterior angle at a vertex of a polygon are a linear pair of angles. Thus if the measure of an interior angle of a regular polygon is 120 , the measure of each exterior angle is 60 . From Corollary 11-2-1, it follows that $\frac{360}{n} = 60$. Therefore $n = 6$.

Find the number of sides of a regular polygon if the measure of each interior angle of the polygon is:

(a) 120 (c) 144
(b) 170 (d) $128\frac{4}{7}$.

6. Complete the following chart:

Name of regular polygon	Sum of the measures of the interior angles	Sum of the measures of exterior angles, one at each vertex	Measure of each interior angle	Measure of each exterior angle
Equilateral Triangle				
Square				
Regular Pentagon				
Regular Hexagon				
Regular Octagon				
Regular Decagon				

7. Consider a regular polygon of twenty sides. Find the measure of:

   (a) Each interior angle of the polygon;
   (b) Each exterior angle of the polygon;
   (c) The sum of the measures of the interior angles of the polygon;
   (d) The sum of the measures of all the exterior angles of the polygon.

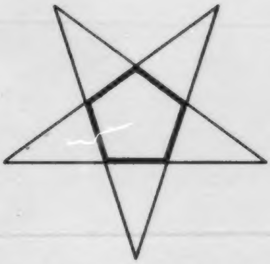
8. In a certain regular polygon, the measure of an exterior angle is one-fifth the measure of an interior angle. Find the number of sides of the polygon.

9. The sum of the measures of eleven angles of a polygon of twelve sides is 1650 .

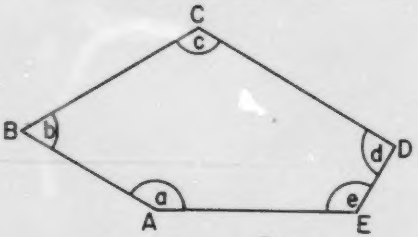
   (a) What is the measure of the remaining angle?
   (b) Do you have enough information to decide whether the polygon is regular? Explain.

10. Is it possible to have a regular polygon with the measure of each interior angle equal to 153 ? Why?

11. The star-shaped figure is formed by extending the sides of a regular pentagon. Find the measure of the angle at each point of the star.

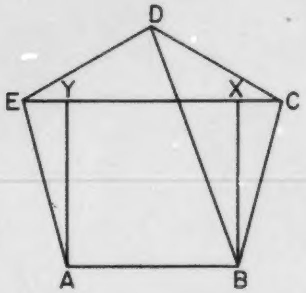


12. Given a pentagon ABCDE such that $m\angle a = 150$, $m\angle b = 60$, and the measures of $\angle c$, $\angle d$, $\angle e$ are proportional to 4 , 3, 4 . Prove that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.



13. In a regular polygon ABCDE...J of at least 5 sides, prove that diagonal $\overline{AD}$ is parallel to side $\overline{BC}$.

14. In the figure, we have given a regular pentagon ABCDE and a rectangle ABXY , where C, X, Y, E are collinear in that order. Find m $\angle$CBX , m $\angle$DCX , and m $\angle$XBD .



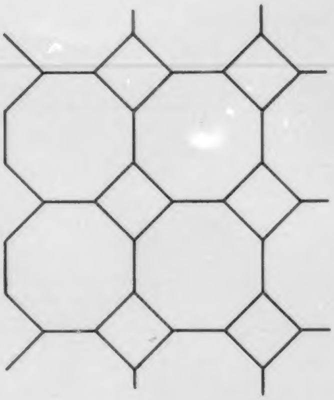
15. Consider the problem of how to cover a polygonal floor with non-overlapping tiles such that any two adjacent tiles have a side in common.

(a) Suppose that each tile has the shape of a square and all tiles are congruent to one another. How many tiles are needed to cover the surface around a point which is at a corner of tiles?

(b) If the tiles are in the shape of congruent equilateral triangles, how many are needed to cover the surface around a point which is at a corner of tiles?

(c) Could the tiles be shaped like other regular polygons of the same number of sides and cover the surface around a point without any overlapping? How many tiles of any one polygonal shape would be needed?

(d) If two tiles have the shape of a regular octagon and another has the shape of a square, the three tiles would cover the surface around a point without overlapping. What other combinations of three regular polygons (two of which are alike) will do this?



Hint: Find solutions of the equation $2x + y = 360$ where x and y are the measures of the interior angles of regular polygons having a different number of sides. In the illustration $x = 135$ and $y = 90$.

(e) Investigate the possibility of other combinations of tiles shaped like regular polygons which would be suitable for use in covering a floor.

16. Consider a sequence of regular polygons with the number of sides as follows: 3, 4, 5, ..., n, ... Choose the expression which correctly completes each of the following sentences.

    (a) The sum of the measures of the interior angles of the polygons (increases, decreases, remains the same.)

    (b) The sum of the measures of the exterior angles, one at each vertex of the polygon, (increases, decreases, remains the same.)

    (c) The measure of an interior angle of the polygon (increases, decreases, remains the same.)

    (d) The measure of an exterior angle of the polygon (increases, decreases, remains the same.)

11-4. Area.

In Chapter 3 we introduced into our formal geometry the notion of the distance between two points. Guided by our experiences from the physical world, we selected postulates and definitions which describe precisely the basic properties of distance in our geometry. We then deduced by logical reasoning other properties of distance and the connections between distance and related topics. In particular, we discussed segments. A segment is a certain set of points; its "size," commonly called its length, we defined to be the same as the distance between its endpoints. The notion of congruence for two segments we described in terms of their lengths.

In a similar manner in Chapter 4, after describing a different type of set of points, namely an angle, we stated, by means of postulates and definitions, exactly what is meant by the measure of an angle. Additional properties of angle measure we deduced as theorems.

We now wish to discuss how to measure a polygonal-region, that is, how to determine its "area." A polygonal-region is a set of points of a quite different type from the segment or the angle. We follow an approach like that used before; namely, we select postulates and definitions which formalize in our geometry the corresponding notion from everyday life. Notice the resemblances between the postulates in this section and those describing distance or angle measure. The first one says that every polygonal-region has a unique measure relative to any standard "unit."

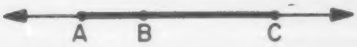
> Postulate 26. If R is any given polygonal-region, there is a correspondence which associates to each polygonal-region in space a unique positive number, such that the number assigned to the given polygonal-region R is one.

DEFINITIONS. The given polygonal-region R mentioned in Postulate 26 is called the unit-area.

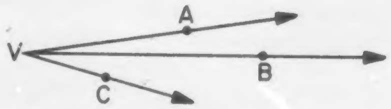
Relative to a given unit-area, the number which corresponds to a polygonal-region, in accordance with Postulate 26, is called the area of the polygonal-region.

Postulate 26 does not tell us what number the area of any particular polygonal-region is (except the unit-area), nor does it tell us how the areas of various polygonal-regions compare. We need more postulates to give us this information.

If a segment is the union of two segments whose interiors do not intersect, then the measure of the given segment is the sum of the measures of the other two segments. In the figure, $AC = AB + BC$.



Recall the corresponding situation for angles, as shown in the diagram, where $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$. The interiors of the two adjacent angles, $\angle AVB$ and $\angle BVC$, do not intersect, and the measure of $\angle AVC$ is the sum of the measures of the two angles $\angle AVB$ and $\angle BVC$. We wish to have a similar property for the areas of polygonal-regions. Thus, for example, if R is the polygonal-region consisting of the parallelogram $ABCD$ and its interior, as shown in Figure a, then we want the area of R to be the sum of the areas of the two triangular-regions R_1 and R_2. The following postulate guarantees this.



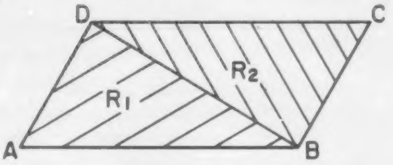


Figure a

Postulate 27. Suppose that the polygonal-region R is the union of two polygonal-regions R_1 and R_2 such that the intersection of R_1 and R_2 is contained in a union of a finite number of segments. Then, relative to a given unit-area, the area of R is the sum of the areas of R_1 and R_2.

In Figure a, the two triangular-regions R_1 and R_2 intersect in a segment. Other illustrations are given in Figures b and c.

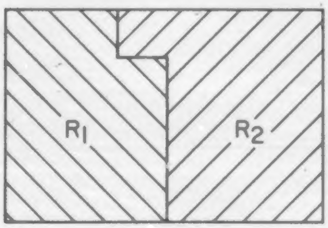


Figure b

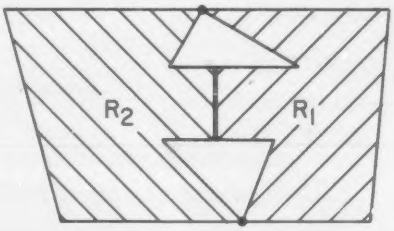


Figure c

In Figure b, the intersection of the polygonal-regions R_1 and R_2 is the union of three segments. In Figure c, the intersection (marked heavily) consists of one segment and two other points; it is contained in the union of a finite number of segments. In each case the sum of the areas of R_1 and R_2 is the area of the entire polygonal-region.

On the other hand, consider the polygonal-region shown in Figure d. It is the union of triangular regions T_1 and T_2 .

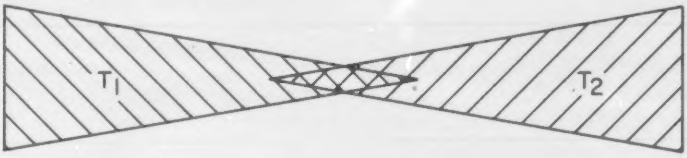
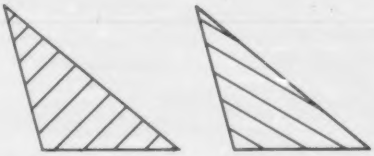


Figure d

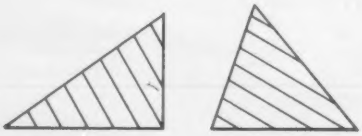
Their intersection is not contained in a union of a finite number of segments, but instead is the cross-hatched polygonal-region whose boundary is a quadrilateral. Thus Postulate 27 is not applicable to this case. If we tried to calculate the area of the entire polygonal-region by adding the areas of T_1 and T_2 , then the area of the polygonal-region which is the intersection would be counted twice. Of course if we cut the entire polygonal-region in a different way, we may be able to apply Postulate 27.

We recall that two segments are congruent if and only if they have the same measure. Two angles are congruent if and only if they have the same measure. We wish to compare the notions of congruence and area for polygonal-regions. Since a polygonal-region is the union of triangular-regions and since we have extensively studied congruence for triangles, we consider triangular-regions in particular. On the basis of our experience in the physical world, two triangular-regions whose respective boundaries are congruent triangles have the "same size and shape." Being of the same "size," their areas seem to be the same. The next postulate guarantees this.



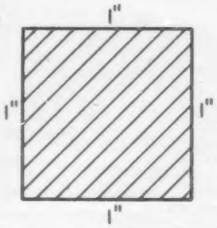
Postulate 28. If two triangles are congruent, then the respective triangular-regions consisting of the triangles and their interiors have the same area relative to any given unit-area.

Thus two triangular-regions with congruent boundaries have the same area. Notice that the converse is not valid. If two triangular-regions have the same area, we do not know whether the triangles which are their respective boundaries are congruent or not. The picture shows two triangles which have different "shapes," although the "sizes," that is, areas, of the corresponding triangular-regions appear to be the same.



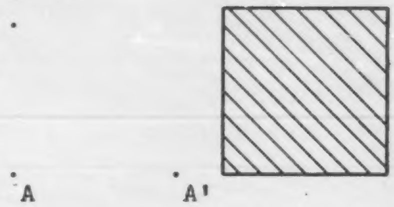
For any convex polygon, the union of the polygon and its interior is a polygonal-region. This polygonal-region has an area relative to a given unit-area. It is customary and very convenient to speak of "the area of the polygon" when we really mean "the area of the associated polygonal-region." Thus, as examples, we speak of the "area of a triangle" when we mean the area of the union of the triangle and its interior; the "area of a parallelogram" is a conveniently short phrase for the "area of the polygonal-region consisting of the parallelogram and its interior."

In the physical world the notion of area is closely related to the notion of distance. If an inch is chosen as a unit of distance, then the customary choice for a unit of area is the "square inch." This is the area of a polygonal-region consisting of a square and its interior such that each side of the square is one inch long. Although some other type of polygonal-region can be chosen as the unit-area in our geometry, we prefer, in this book, to adopt as our unit-area the so-called "unit-square," which is defined as follows:



DEFINITION. Given a unit-pair for measuring distance, a unit-area is called a unit-square if and only if the unit-area consists of a square and its interior such that the measure of a side of the square is one.

The diagram pictures a unit-square relative to the unit-pair {A,A'} .



Our fourth postulate concerning area tells us how to determine the area of certain polygonal-regions. It connects the concept of area with the concept of distance developed in Chapter 3.

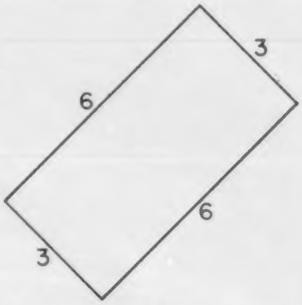
Postulate 29. Given a unit-pair for measuring distance, the area of a rectangle relative to a unit-square is the product of the measures (relative to the given unit-pair) of any two consecutive sides of the rectangle.

DEFINITIONS. Any side of a parallelogram is a base of the parallelogram.

An altitude of the parallelogram relative to the base is any segment which is perpendicular to the base and whose respective endpoints lie on the parallel lines containing the base and the side opposite to the base.

In particular, any side of a rectangle is a base of the rectangle, and any side which is consecutive to the base of the rectangle is an altitude of the rectangle (relative to the base).

Our work in the following sections is largely concerned with the areas of certain polygonal-regions and the lengths of certain segments related to the polygonal-regions. It is customary to shorten the phrase "the length of a side" of a polygon and say simply "the side," whenever the context makes clear that we mean a number rather than a segment. In a like manner, a base of a parallelogram or a diagonal of a polygon is a segment, that is, a set of points; sometimes, however, we use the word "base" or "diagonal" to mean the number which is the length of the segment; we do this only in case there is no danger of confusion between the two different uses of the same word.



If the lengths of two consecutive sides of a rectangle are 6 and 3 , then we may consider the base to be 6 and the altitude 3 ; or we may choose 3 as the base, in which case 6 is the altitude. For either choice, the area of the rectangle is 18 .

Using the terminology given by the last definitions, Postulate 29 tells us that:

> The area of a rectangle is the product of its base and its altitude.

If the area, the base, and the altitude of a rectangle are denoted by A, b, h, respectively, then

$$A = bh .$$

As a special case, the area A of a square each of whose sides has length s is given by

$$A = s^2 .$$

Problem Set 11-4

1. Complete each of the following tables and answer the questions pertaining to each.

(a)

▭	Base	Altitude	Area
a	3	2	6
b	6	2	
c	12	2	
d	24	2	

Consider a set of rectangles with equal altitudes. If these rectangles are arranged so that the bases of any two consecutive rectangles have the ratio of 1 to 2 , then the ratio of the areas of any two consecutive rectangles is ______ to ____.

(b)

▭	Base	Altitude	Area
a	2	3	6
b	2	9	
c	2	27	

Consider a set of rectangles with equal bases. If these rectangles are arranged so that the altitudes of any two consecutive rectangles have the ratio of 1 to 3 , then the ratio of the areas of any two consecutive rectangles is _____ to _____.

(c)

▭	Base	Altitude	Area
a	5		100
b	10		100
c	20		100
d	40		100
e	80		100

Consider a set of rectangles with equal areas. If these rectangles are arranged so that the bases of any two consecutive rectangles have the ratio of 1 to 2 , then the ratio of the altitudes of any two consecutive rectangles is _____ to _____.

(d)

▭	Base	Altitude	Area
a	1	2	
b	3	6	
c	9	18	
d	27	54	

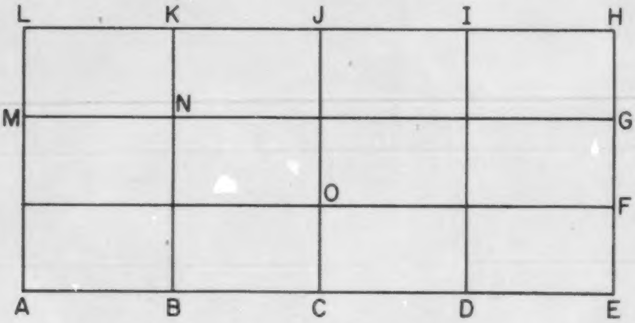
What is the ratio of the bases of any two consecutive rectangles in the table? What is the ratio of the corresponding altitudes? What is the ratio of the corresponding areas? The four rectangles are members of a set of ________ rectangles.

(e) Complete the following sentences:

If the ratio of the lengths of a pair of corresponding sides of two similar rectangles is 1 to 3 , the ratio of the areas is _____ to _____.

If the ratio of the lengths of a pair of corresponding sides of two similar rectangles is 2 to 3 , the ratio of the areas is _____ to _____.

2.



This figure is separated into twelve rectangular regions. Let each small region be k units long and one unit high.

What is the ratio of the areas of each of the following pairs of rectangles?

(a) Rectangle AN to rectangle AK . (Here we name a rectangle by naming a pair of opposite vertices.)

(b) Rectangle AJ to rectangle AH .

(c) Rectangle AO to rectangle AF .

(d) Rectangle BI to rectangle CI .

(e) Rectangle BF to rectangle CF .

(f) Rectangle BO to rectangle ND .

3. (a) We are given two rectangles with equal bases. If the ratio of the altitudes is 1 to 3 , the ratio of the areas is ______ to ______.

(b) If the bases of two rectangles are in the ratio of 1 to 4 and the corresponding altitudes are in the ratio of 1 to 2 , the ratio of the areas of the rectangles is ______ to ______.

(c) If the areas of two rectangles are equal and the ratio of the bases is 1 to 3 , then the ratio of the altitudes is ______ to ______.

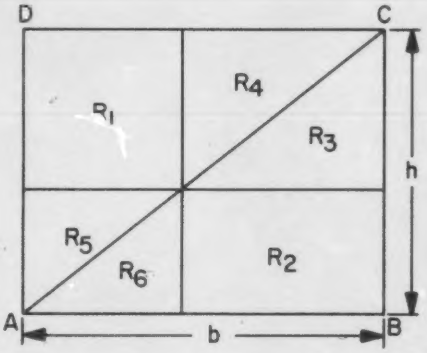
(d) If the bases of two rectangles are equal, and the altitude of the second is 25 per cent more than the altitude of the first, then the ratio of the areas of the first to the second is ______ to ______.

4. The ratio of the lengths of two consecutive sides of a rectangle is 4 to 5 . If the area of the rectangle is 5780 , find the length of each side.

5. Let a and b be positive numbers. Show by a drawing that the area of a square whose side measures $a + b$ is the same as the sum of the areas of:

   (a) a square whose side measures a ,
   (b) a square whose side measures b , and
   (c) two rectangles each of which has sides measuring a and b .

6. In the figure, $\overline{AC}$ is a diagonal of rectangle ABCD . The polygonal-region ABCD is cut into 6 polygonal-regions: the boundary of R_1 is a square; the boundary of R_2 is a rectangle; R_3, R_4, R_5, R_6 are triangular regions.



   (a) The area of $\triangle ABC$ is the sum of the areas of the polygonal-regions _____, _____, _____.

   (b) The area of $\triangle ADC$ is the sum of the areas of the polygonal-regions _____, _____, _____.

   (c) The area of $\triangle ABC$ is equal to the area of $\triangle ADC$. Why?

   (d) The areas of R_5 and R_6 are equal. Why?

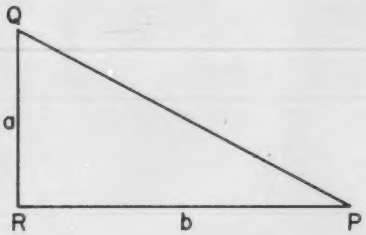
   (e) The areas of R_3 and R_4 are equal. Why?

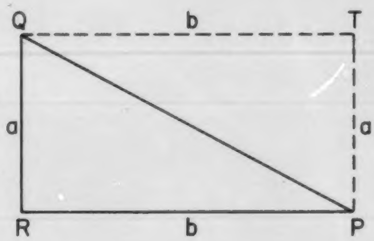
   (f) Therefore, the areas of R_1 and R_2 are equal. Why?

11-5. Areas of Triangles and Quadrilaterals.

On the basis of the four postulates concerning area in the preceding section, we can calculate the areas of triangles, parallelograms, and various other quadrilaterals.

THEOREM 11-3. The area of a right triangle is one half the product of the lengths of its two legs.





Proof: Let triangle PQR have a right angle at R . Let the lengths of its legs be a and b , and let A be the area of the triangle. (The diagram above shows two pictures of the same triangle PQR .) Let T be the intersection of the line parallel to $\overleftrightarrow{PR}$ through Q and the line parallel to $\overleftrightarrow{QR}$ through P . Then QTPR is a rectangle, and $\triangle PQR \cong \triangle QPT$. By Postulate 28, the area of $\triangle QPT$ is A . By Postulate 27, the area of the rectangle QTPR is $A + A$, because the two triangular regions intersect only in the segment $\overline{PQ}$. By Postulate 29, the area of the rectangle is ab . Therefore

$$2A = ab ,$$

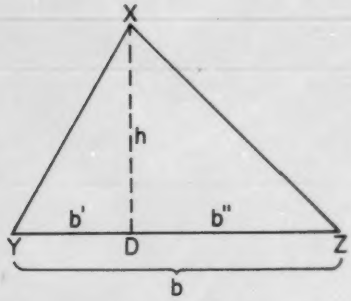
or

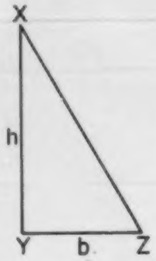
$$A = \frac{1}{2}ab .$$

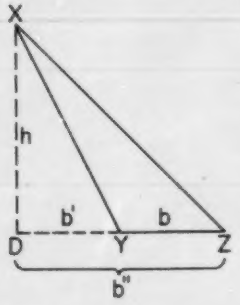
From this we can derive the formula for the area of any triangle. Once we obtain this formula, it will include Theorem 11-3 as a special case.

THEOREM 11-4. The area of a triangle is one-half the product of any base and the altitude to that base.

Proof: Let A be the area of the given triangle XYZ. Consider the altitude $\overline{XD}$ to the side $\overline{YZ}$ of the triangle. Let $b = YZ$ and $h = XD$. Let the distances, b' and b'', between D and the endpoints of the side opposite X be chosen so that $b' \leq b''$. There are three cases to consider.







(1) If D is between Y and Z, then $\overline{XD}$ cuts the given triangle into two right triangles, with bases b' and b'', as indicated. Furthermore, $b = b' + b''$. By the preceding theorem, these two right triangles have respective areas $\frac{1}{2}b'h$ and $\frac{1}{2}b''h$.

Hence, by Postulate 27,

$$\begin{aligned} A &= \frac{1}{2}b'h + \frac{1}{2}b''h \\ &= \frac{1}{2}(b' + b'')h \\ &= \frac{1}{2}bh \, . \end{aligned}$$

(2) If D is one of the endpoints of $\overline{YZ}$, then $\triangle XYZ$ is a right triangle. Therefore, $A = \frac{1}{2}bh$, by Theorem 11-3.

(3) If D is not on the segment $\overline{YZ}$, there are again two right triangles, namely $\triangle XDZ$ and $\triangle XDY$. In this case, $b' + b = b''$. Hence

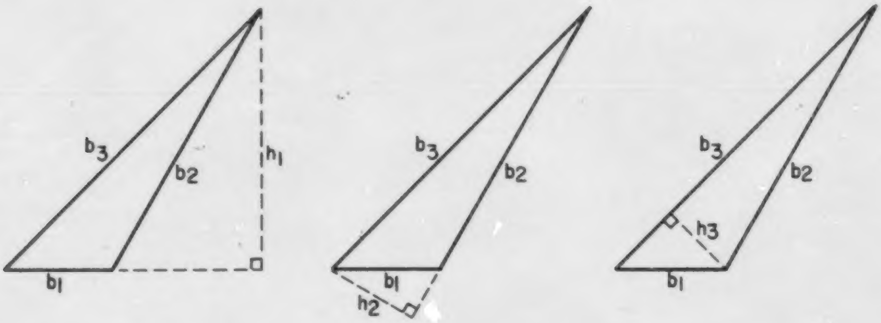
$$\frac{1}{2}b'h + A = \frac{1}{2}(b' + b)h \ .$$

Why? Solving the above equation for A , we obtain

$$A = \frac{1}{2}bh \ .$$

Explain how.

Since the length of any side of a triangle can be chosen as the base, Theorem 11-4 can be applied to any triangle in three different ways. The figure below shows the three choices for a single triangle. Any of the three formulas, $A = \frac{1}{2}b_1h_1$, $A = \frac{1}{2}b_2h_2$, $A = \frac{1}{2}b_3h_3$, gives the area of the triangle.



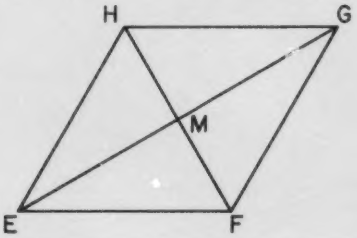
Corollary 11-4-1. The area A of an equilateral triangle whose side has length s is given by:

$$A = \frac{\sqrt{3}}{4}s^2 \ .$$

The proof is left as a problem.

THEOREM 11-5. The area of a rhombus is one half the product of the lengths of the diagonals.

Proof: Let M be the point of intersection of the diagonals of the rhombus EFGH, namely $\overline{EG}$ and $\overline{FH}$.



Let $d = EG$ and $d' = FH$. The diagonals are perpendicular to each other. Since $\overline{FM}$ is the altitude to side $\overline{EG}$ of triangle EFG, the area of $\triangle EFG$ is

$$\frac{1}{2}d(FM) .$$

In a like manner, we note that the area of $\triangle EGH$ is

$$\frac{1}{2}d(HM) .$$

Hence, by Postulate 27, the area of the rhombus is

$$\frac{1}{2}d(FM) + \frac{1}{2}d(HM) = \frac{1}{2}d(FH) = \frac{1}{2}d\ d' .$$

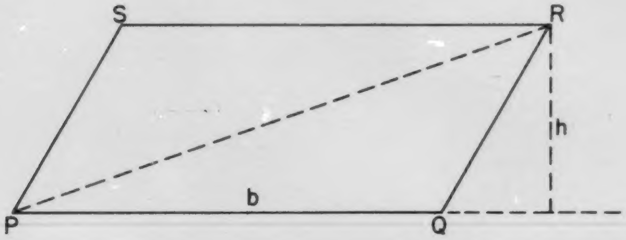
Corollary 11-5-1. The area A of the square whose diagonal has length d is given by

$$A = \frac{1}{2}d^2 .$$

The proof is left as a problem.

THEOREM 11-6. The area of a parallelogram is the product of any base and the altitude to that base.

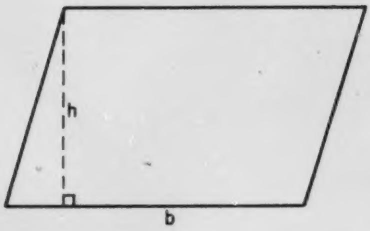
Proof: Let A be the area of the parallelogram PQRS. Let b and h be a base and the corresponding altitude.

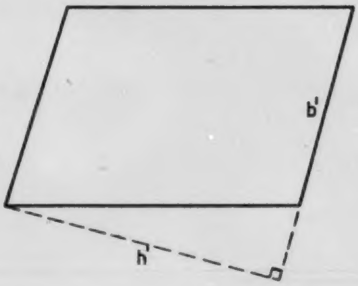


The triangles PQR and RSP, which have the diagonal $\overline{PR}$ of the parallelogram as a common side, are congruent. Hence the triangular-regions PQR and RSP have the same area, by Postulate 28. Hence the area of PQRS is twice the area of $\triangle PQR$. Since b and h are a base and a corresponding altitude of $\triangle PQR$, the area of $\triangle PQR$ is $\frac{1}{2}bh$. Therefore the area of PQRS is $2(\frac{1}{2}bh)$, or

$$A = bh .$$

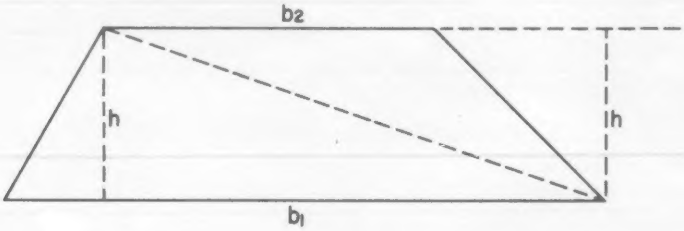
Since the length of any side of a parallelogram can be taken as the base, Theorem 11-6 can be applied to any parallelogram in two ways. The figures following illustrates the two choices for a single parallelogram. In one case, we obtain $A = bh$, and in the other, $A = b'h'$. Either of these two expressions gives the area of the parallelogram.





THEOREM 11-7. The area of a trapezoid is one-half the product of its altitude and the sum of its bases.

Proof: Let A be the area of the trapezoid, h its altitude, and b_1 and b_2 its bases.



A diagonal of the trapezoid cuts the polygonal-region into two triangular-regions whose respective areas are $\frac{1}{2}b_1h$ and $\frac{1}{2}b_2h$. (The dotted lines on the right in the diagram indicate why the two triangles have the same altitude.) By Postulate 27, the area of the trapezoid is

$$A = \frac{1}{2}b_1h + \frac{1}{2}b_2h \ .$$

Algebraically, this is equivalent to the formula

$$A = \frac{1}{2}h(b_1 + b_2) \ .$$

DEFINITION. The median of a trapezoid is the segment which joins the midpoints of the two non-parallel sides.

Corollary 11-7-1. The area of a trapezoid is equal to the product of its altitude and the length of its median.

The proof is left as a problem.

Summary of Formulas:

Area of a rectangle: $A = bh$.

Area of a parallelogram: $A = bh$.

Area of a triangle: $A = \frac{bh}{2}$.

Area of an equilateral triangle: $A = \frac{\sqrt{3}\, s^2}{4}$.

Area of a rhombus: $A = \frac{dd'}{2}$.

Area of a square: $A = s^2$, $A = \frac{d^2}{2}$.

Area of a trapezoid: $A = \frac{h(b_1 + b_2)}{2}$.

$A = hm$.

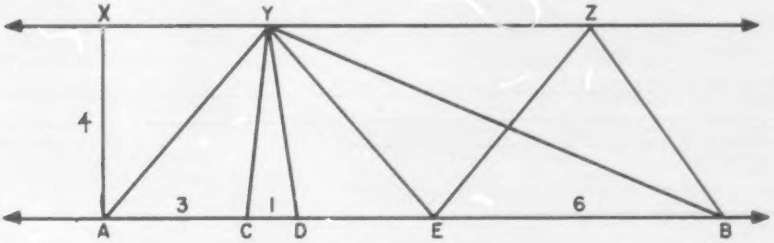
Problem Set 11-5

1. Find the area of a right triangle if the lengths of the legs of the triangle are 6 and 10 .
2. Find the area of an isosceles right triangle if the length of each of the congruent sides of the triangle is 12 .
3. Find the area of a 45-45-90 triangle if the hypotenuse of the triangle is 12 .
4. Find the area of a 30-60-90 triangle if the hypotenuse of the triangle is 12 .
5. If h is the hypotenuse of a 45-45-90 triangle, find:
   - (a) The length in terms of h of the side opposite an angle whose measure is 45 .
   - (b) The area of the triangle in terms of h .
6. If h is the hypotenuse of a 30-60-90 triangle, find:
   - (a) The length in terms of h of the side opposite the angle with measure of 30 .
   - (b) The length in terms of h of the side opposite the angle with measure 60 .
   - (c) The area of the triangle in terms of h .

7. Find the unknown in each of the following triangles if A is the area, b the base, and h the altitude.

	A	b	h
(a)	?	12	10
(b)	?	6	8
(c)	12	8	?
(d)	12	?	6

8. In the diagram, the line containing A, C, D, E, B is parallel to the line containing X, Y, Z, and is perpendicular to $\overleftrightarrow{AX}$. $AC = 3$; $CD = 1$; $EB = 6$; $AX = 4$.



In each of the following, find the ratio of the area of the first-named triangle to the area of the second-named triangle.

(a) $\triangle AYC$; $\triangle AYD$

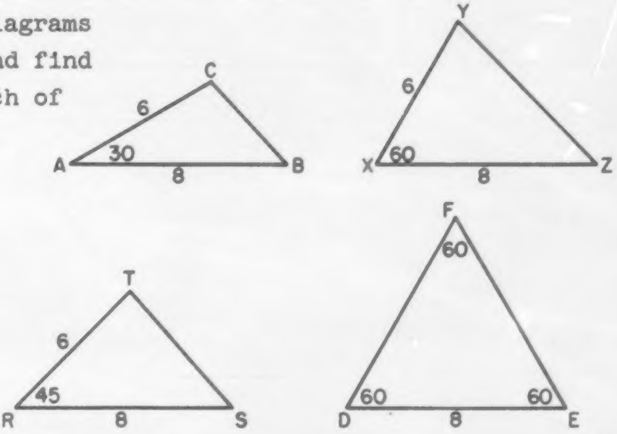
(c) $\triangle AYC$; $\triangle EZB$

(b) $\triangle AYC$; $\triangle EYB$

(d) $\triangle EYB$; $\triangle EZB$

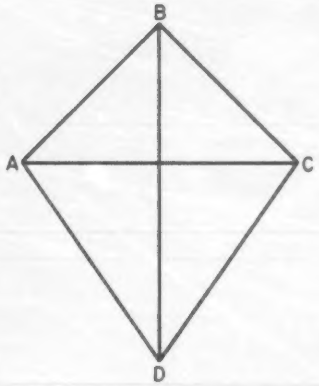
9. Refer to the diagrams at the right and find the area of each of the following:

(a) $\triangle ABC$
(b) $\triangle XYZ$
(c) $\triangle RST$
(d) $\triangle DEF$



10. Prove Corollary 11-4-1.

11. In the diagram to the right, ABCD is a quadrilateral with diagonals $\overline{AC}$ and $\overline{BD}$ perpendicular to each other. $\overline{BD}$ bisects $\overline{AC}$. $AC = 20$ and $BD = 24$. Find the area of the quadrilateral.



12. If ABCD is a rhombus with diagonals 20 and 24 , find the area of the rhombus.

13. Find the area of a rhombus if the length of one side of the rhombus is 15 and the longer diagonal of the rhombus is 24 .

14. The area of a rhombus is 1600 . Find the length of each diagonal of the rhombus if one is twice as long as the other.

15. Prove Corollary 11-5-1.

16. Find the area of a square if the diagonal of the square is 8 .

17. Find the area of a parallelogram if the base of the parallelogram is 12 and the altitude of the parallelogram is 7 .

18. The area of a parallelogram is 8430 and the altitude of the parallelogram is 150 . Find the base.

19. Find the area of a parallelogram ABCD if $AB = 10$ and $AD = 14$, and:

(a) $m\angle A = 30$.
(b) $m\angle B = 45$.
(c) $m\angle A = 60$.
(d) $m\angle D = 90$.

20. The sides of a parallelogram are 8 and 10 respectively. If the shorter altitude is 4 , what is the longer altitude?

21. Find the unknown in each of the following trapezoids if A is the area, h the altitude, and b_1 and b_2 the bases of the trapezoid.

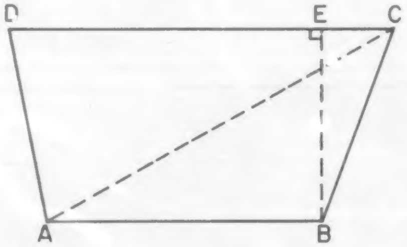
	A	h	b_1	b_2
(a)	?	3	6	8
(b)	?	6	24	10
(c)	72	?	5	7
(d)	100	5	4	?
(e)	180	?	11	9

22. ABCD is an isosceles trapezoid with $\overline{AB} \parallel \overline{DC}$ and $m\angle A = 30$. AB = 10 and DC = 6 . Find the area of the trapezoid.

23. Prove Corollary 11-7-1.

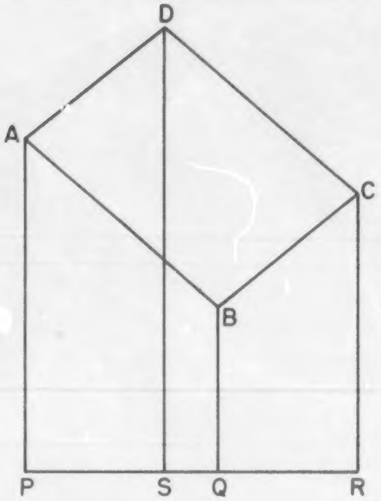
24. Find the side of a square if the area of the square is equal to the area of a rectangle 16 feet by 9 feet.

25. In quadrilateral ABCD , $\overline{DC} \parallel \overline{AB}$ and $\overline{BE} \perp \overline{DC}$. If AB = 10 , DC = 14 , EB = 7 , find the areas of $\triangle ADC$ and $\triangle ABC$.



26. The points P, S, Q, R, collinear in that order, and the parallelogram ABCD lie in the same plane; and $\overline{AP}$, $\overline{BQ}$, $\overline{CR}$ and $\overline{DS}$ are perpendicular to $\overline{PR}$. AP = 12, BQ = 6, DS = 16, QR = 5, CR = 10, SQ = 2, PS = 5.

Find the area of ABCD.



27. The vertices of a triangle have coordinates (-2,-3), (-4,1), and (4,5). Prove that the triangle is a right triangle. Find the area of the triangle.

28. Three of the vertices of a rhombus ABCD are: A(0,0), B(-6,-2), C(-8,-8).

(a) What are the coordinates of vertex D ?

(b) Find the area of the rhombus.

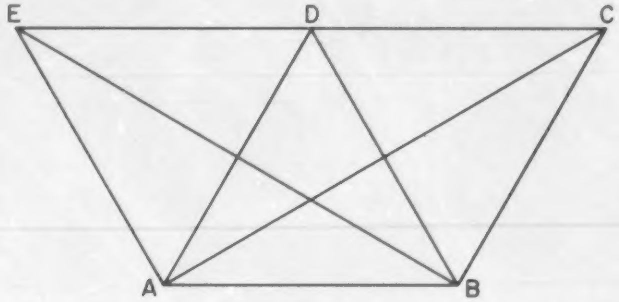
29. The vertices of a trapezoid have the following coordinates: A(0,0), B(12,0), C(17,6) and D(2,6). Find the altitude and the area of the trapezoid.

30. The vertices of a quadrilateral ABCD have the following coordinates: A(-3,0), B(2,4), C(6,0), and D(3,-5). Find the area of the quadrilateral. Hint: Consider the altitudes of $\triangle ABC$ and $\triangle ADC$.

31. The coordinates of the respective vertices of rectangle ABCD are (3,2), (10,2), (10,7) and (3,7). In the same coordinate system, the vertices of $\triangle EFC$ are (5,2), (3,5), and (10,7). Find the area of $\triangle EFC$.

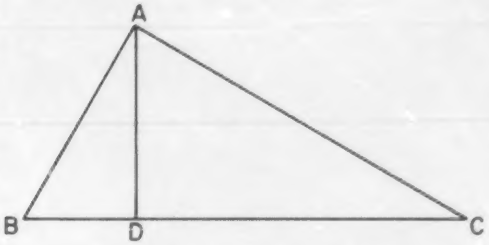
32. In the quadrilateral ABCE , point D is between C and E ; $\overleftrightarrow{EC} \parallel \overleftrightarrow{AB}$; AB = BC = CD = DE = EA .

Prove: AC · BD = EB · AD .

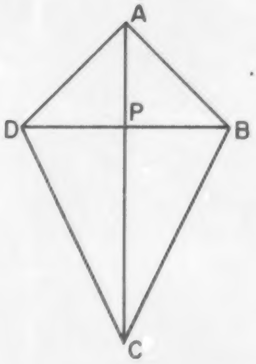


33. The hypotenuse of right triangle ABC is $\overline{BC}$, and $\overline{AD}$ is the altitude to the hypotenuse.

Prove: AB · AC = BC · AD .



34. Prove: If the diagonals of a quadrilateral are perpendicular, the area of the quadrilateral is equal to one-half the product of the lengths of the diagonals.



11-6. Area Relations.

In the study of area it is interesting and important to compare the areas of two or more figures when they differ in one or more dimensions. Proportionality is one of the most effective methods of studying this change. Before continuing, you may wish to review the definition and fundamental properties of proportionality and of proportions, as presented in Chapter 7.

Consider two triangles. Suppose that one of the triangles has base b_1, altitude h_1, and area A_1; suppose that the other triangle has base b_2, altitude h_2, and area A_2. Then

$$A_1 = \frac{1}{2}b_1h_1$$

and

$$A_2 = \frac{1}{2}b_2h_2 .$$

Hence, by division,

$$\frac{A_2}{A_1} = \frac{b_2h_2}{b_1h_1} .$$

If the two triangles have the property that $b_1 = b_2$, then $\frac{A_2}{A_1} = \frac{h_2}{h_1}$. In other words, the areas of two triangles with equal bases are proportional to the corresponding altitudes. We note that $A_2 = k\,h_2$ and $A_1 = k\,h_1$, where the constant of proportionality k is one-half the base of each triangle, namely $k = \frac{1}{2}b_1 = \frac{1}{2}b_2$.

If the two triangles under consideration have the property $h_1 = h_2$, then

$$\frac{A_2}{A_1} = \frac{b_2}{b_1} .$$

In other words, the areas of the two triangles are proportional to the bases.

On the other hand, if the two triangles have the property that $A_1 = A_2$, then $b_1h_1 = b_2h_2$. In other words, the product of the base and altitude is the same for one triangle as for the other. This situation suggests a notion which is related to the concept of proportionality and which we wish to discuss now, namely "inverse proportionality."

An important property of proportionality is the following, expressed for the case of three numbers: If the positive numbers q, r, s are proportional to the positive numbers a, b, c then the largest of the numbers a, b, c corresponds to the largest of the numbers q, r, s . By contrast, as the definition below shows, if the positive numbers q, r, s are inversely proportional to the positive numbers a, b, c , then the largest of a, b, c corresponds to the smallest of the numbers q, r, s . With this introduction, we are ready for the definition.

DEFINITION. Suppose that to the positive numbers $q, r, s, \ldots$ there correspond the positive numbers $a, b, c, \ldots$ (that is, $q \longleftrightarrow a$, $r \longleftrightarrow b$, $s \longleftrightarrow c$, ...). The numbers $q, r, s, \ldots$ are inversely proportional to the numbers $a, b, c, \ldots$ if and only if all the products of corresponding numbers are the same (that is, $qa = rb = sc = \ldots$).

As an example, the numbers 2 , 6 , 15 , 12 are inversely proportional to the numbers 9 , 3 , $\frac{6}{5}$, $\frac{3}{2}$, because each product of corresponding numbers is 18 .

As another example, find the numbers x and y such that 2 , x , 5 are inversely proportional to 6 , 4 , y . By the definition, the products $2 \cdot 6$, $x \cdot 4$, and $5 \cdot y$ are all the same. Thus $12 = 4x = 5y$. The desired numbers are $x = 3$ and $y = \frac{12}{5}$.

We now extend our preliminary remarks about two triangles to the case of any number of triangles.

THEOREM 11-8. Consider a set of two or more triangles.

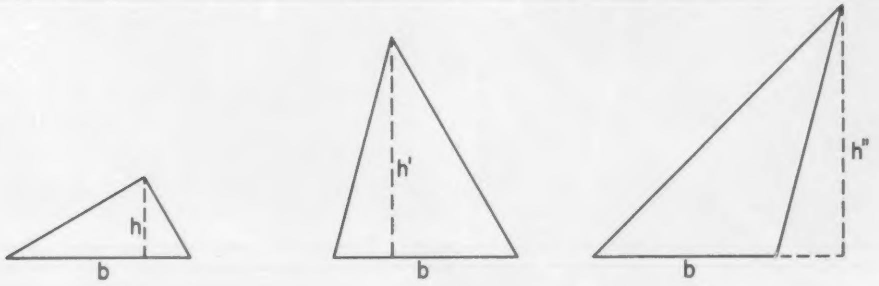
(a) If the bases of all the triangles are equal, then the areas of the triangles are proportional to the corresponding altitudes.

(b) If the altitudes of all the triangles are equal, then the areas of the triangles are proportional to the corresponding bases.

(c) If the areas of all the triangles are equal, then the bases of the triangles are inversely proportional to the corresponding altitudes.

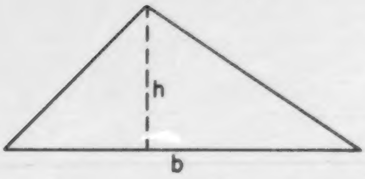
Proof: For definiteness, we prove the theorem for a set of three triangles; the method applies to any number; by choosing three, we avoid complications of notation in discussing many triangles.

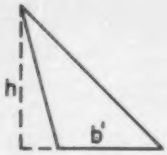
(a) By hypothesis, all the bases are the same number, say b . Let the areas of the triangles be A , A' , A'' , and let the corresponding altitudes be h , h' , h'' .

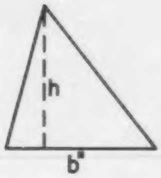


Now $A = \frac{b}{2}h$, $A' = \frac{b}{2}h'$, $A'' = \frac{b}{2}h''$. Hence the numbers A , A' , A'' are proportional to the numbers h , h' , h'' , with the non-zero number $\frac{b}{2}$ as the proportionality constant.

(b) By hypothesis, all the altitudes are the same number, say h . Let the areas of the triangles be A , A' , A'' and let the corresponding bases be b , b' , b'' .

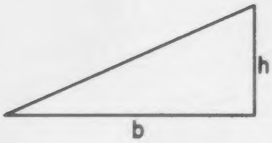


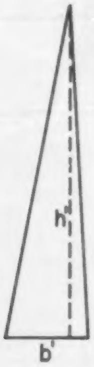


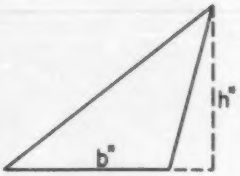


Now $A = \frac{h}{2}b$, $A' = \frac{h}{2}b'$, $A'' = \frac{h}{2}b''$. Thus A , A' , A'' are proportional to b , b' , b'' , with the non-zero number $\frac{h}{2}$ as the proportionality constant.

(c) By hypothesis, all the areas are the same number, say A . Let the bases of the triangles be b , b' , b'' and let the corresponding altitudes be h , h' , h'' .







Now $A = \frac{1}{2}bh$, $A = \frac{1}{2}b'h'$, $A = \frac{1}{2}b''h''$. That is, all of the products bh , $b'h'$, $b''h''$ are equal, since each of them is equal to $2A$. Thus, b , b' , b'' are inversely proportional to h , h' , h'' .

Analogous to Theorem 11-8 is the following theorem for parallelograms.

THEOREM 11-9. Consider a set of two or more parallelograms.

(a) If the bases of all the parallelograms are equal, then the areas of the parallelograms are proportional to the corresponding altitudes.

(b) If the altitudes of all the parallelograms are equal, then the areas of the parallelograms are proportional to the corresponding bases.

(c) If the areas of all the parallelograms are equal, then the bases of the parallelograms are inversely proportional to the corresponding altitudes.

The proof is left as a problem.

A special case of Theorems 11-8 and 11-9 occurs when the number of triangles or parallelograms is two. In fact the case of two triangles has already been mentioned. Nevertheless it is worthy of repetition. If b, h, A are the base, altitude, area, respectively, of one triangle or parallelogram and if b', h', A' pertain to the other, then the respective parts of the two theorems tell us the following:

(a) If $b = b'$, then A, A' are proportional to h, h', and hence $\frac{A}{A'} = \frac{h}{h'}$.

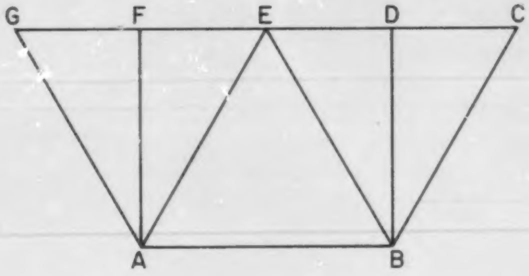
(b) If $h = h'$, then $\frac{A}{A'} = \frac{b}{b'}$.

(c) If $A = A'$, then $bh = b'h'$.

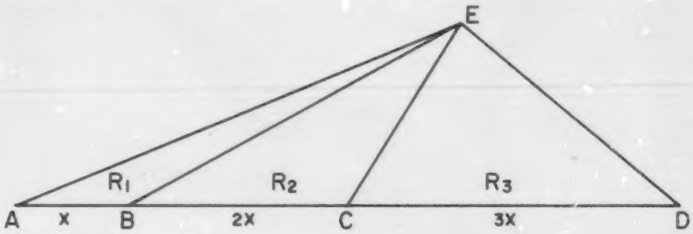
Problem Set 11-6

1. Prove Theorem 11-9.

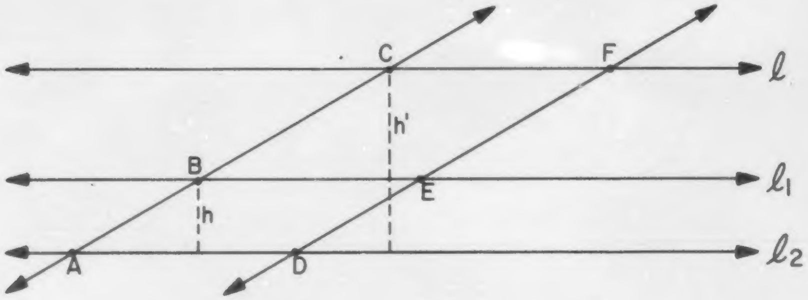
2. In the quadrilateral ABCG , ABDF is a rectangle, and ABCE and ABEG are parallelograms. Compare the areas of the three parallelograms. Explain your answer.



3. As shown in the figure, $\overline{AD}$ is divided into three segments whose measures are proportional to 1 , 2 , 3 . Compare the areas of the three triangular-regions R_1 , R_2 , R_3 .



4. It is given that $\ell \parallel \ell_1 \parallel \ell_2$; $\overleftrightarrow{AC} \parallel \overleftrightarrow{DF}$; m $\angle CAD = 30$; AB = 4x ; BC = 6x . What is the ratio of the areas of ADEB and ADFC ?



5. Prove: The diagonals of a parallelogram divide the parallelogram and its interior into four triangular-regions of equal area.

6. Prove that each median of a triangle cuts the triangular-region into two triangular-regions of equal area.

7. $\overline{AE}$, $\overline{CD}$, and $\overline{BF}$ are medians of $\triangle ABC$ intersecting at point O . Prove that the areas of $\triangle AOB$, $\triangle BOC$ and $\triangle COA$ are equal. Hint: Use Problem 6 to compare the areas of:

   (a) $\triangle ACD$ and $\triangle DCB$; $\triangle ABF$ and $\triangle FBC$.

   (b) $\triangle AOD$ and $\triangle DOB$; $\triangle BOE$ and $\triangle EOC$.

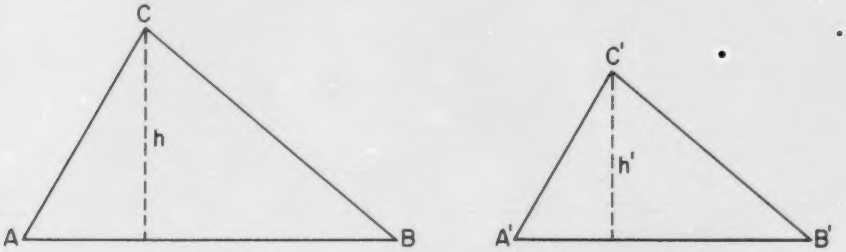
   Then prove that the areas of $\triangle AOB$, $\triangle BOC$, and $\triangle COA$ are equal.

8. The following experiment illustrates the fact that the point of intersection of the medians of a physical triangle is the center of gravity of the triangle.

   Cut a model of a triangle from cardboard and draw the three medians of the triangle. Try to balance the triangle on the head of a pin at the point of intersection of the medians. Use the results in Problem 7 to explain why the intersection of the medians is the balance point or center of gravity.

9. If the area of $\triangle ABC$ in Problem 7 is 216 , find the area of each of the following triangles: $\triangle ABO$, $\triangle BOC$, $\triangle AOC$, $\triangle ODB$, $\triangle BOE$, $\triangle AOF$.

*10. Given: $\triangle ABC \sim \triangle A'B'C'$.



(a) If $AB = 12$, $A'B' = 7$, and $h = 10$, find h' .

(b) If $h = 9$, $h' = 3$, and $A'B' = 4$, find the length of $\overline{AB}$.

(c) If $AB = 7$, $BC = 8$, $AC = 6$, and $A'B' = 3\frac{1}{2}$, find the perimeter of $\triangle A'B'C'$.

11. In $\triangle ABC$, $\overline{CG} \perp \overline{AB}$; $\overline{DF} \parallel \overline{AB}$; $CF = 20$; $CE = 10$; $CB = 30$, $DF = 18$.

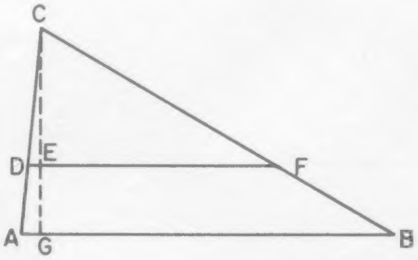
Find:

(a) The length of $\overline{CG}$;

(b) The area of $\triangle DFC$;

(c) The area of $\triangle ABC$;

(d) The area of quadrilateral ABFD .



12. The areas of two triangles are equal. What is the ratio of the base of the first to a base of the second if the corresponding altitude of the second is:

(a) Three times the corresponding altitude of the first.

(b) One-fourth the corresponding altitude of the first.

(c) Three-fourths the corresponding altitude of the first.

(d) One hundred fifty per cent of the corresponding altitude of the first.

(e) Ten per cent more than the corresponding altitude of the first.

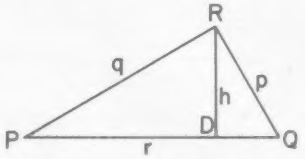
13. Are the areas of two triangles equal if a base of the second is 5 units more than a base of the first, and the corresponding altitude of the second is 5 units less than the corresponding altitude of the first? Explain your answer.

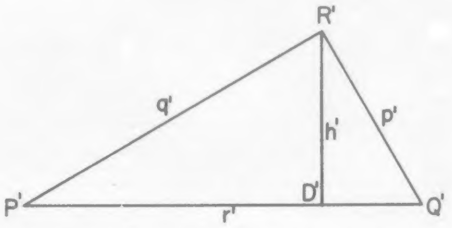
14. What is the ratio of the areas of two rectangles if the base of the second is 25 per cent more than the base of the first, and the altitude of the second is 25 per cent less than the altitude of the first?

11-7. Relations in Similar Polygons.

THEOREM 11-10. Every similarity between triangles has the property that the measures of the three sides and any altitude of the one triangle are proportional to the measures of the corresponding sides and the corresponding altitude of the other triangle.

Proof: Let $PQR \longleftrightarrow P'Q'R'$ be a similarity between triangles. Let k be the proportionality constant. Then





$p = kp'$, $q = kq'$, $r = kr'$. Let $\overline{RD}$ and $\overline{R'D'}$ be the respective altitudes from R in $\triangle PQR$ and from R' in $\triangle P'Q'R'$. Let $h = RD$ and $h' = R'D'$.

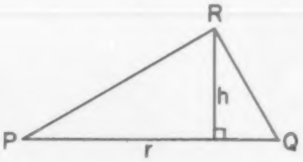
If $D \neq Q$, then consider the correspondence $RDQ \longleftrightarrow R'D'Q'$ between right triangles. Since $\angle RDQ \cong \angle R'D'Q'$ (why?) and $\angle DQR \cong \angle D'Q'R'$, the correspondence is a similarity. The proportionality constant for the similarity $RDQ \longleftrightarrow R'D'Q'$ is also k , since $RQ = p = kp' = k \cdot R'Q'$. Hence $h = kh'$.

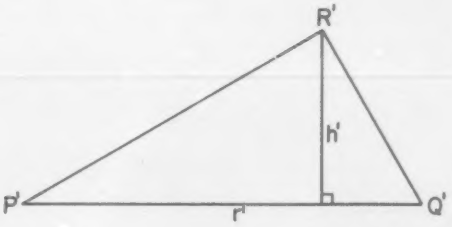
On the other hand, if $D = Q$, then $h = RD = RQ = p$ and $h' = p'$; in this case also, $h = kh'$, since $p = kp'$.

Thus, in every case, p , q , r , h are proportional to p' , q' , r' , h' , with proportionality constant k .

THEOREM 11-11. Every similarity between triangles has the property that the areas of the triangles are proportional to the squares of the lengths of any pair of corresponding sides.

Proof: Let $PQR \leftrightarrow P'Q'R'$ be a similarity between triangles. Consider any pair of corresponding sides, say $\overline{PQ}$ and $\overline{P'Q'}$ and let r and r' be the respective lengths of these sides. Let h and h' be the lengths of the altitudes to these sides in the respective triangles.





Let A and A' be the respective areas of $\triangle PQR$ and $\triangle P'Q'R'$. By Theorem 11-10, $(r,h) \underset{p}{=} (r',h')$. Thus

$$\frac{r}{r'} = \frac{h}{h'} .$$

Now

$$\frac{A}{A'} = \frac{\frac{1}{2}r\ h}{\frac{1}{2}r'h'} = \left(\frac{r}{r'}\right)\left(\frac{h}{h'}\right) .$$

By substitution,

$$\frac{A}{A'} = \left(\frac{r}{r'}\right)\left(\frac{r}{r'}\right) = \frac{r^2}{r'^2} .$$

Thus, $(A,A') \underset{p}{=} (r^2,r'^2)$, as asserted.

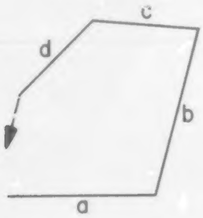
As an example, suppose that $DEF \leftrightarrow LMN$ is a similarity between triangles such that an altitude of $\triangle DEF$ is three times as long as the corresponding altitude of $\triangle LMN$. Then, by Theorem 11-10, every side of $\triangle DEF$ is three times as long as the corresponding side of $\triangle LMN$, and every altitude of $\triangle DEF$ is three times as long as the corresponding altitude of $\triangle LMN$. By addition, the perimeter of $\triangle DEF$ is three times the perimeter of $\triangle LMN$. Furthermore, by Theorem 11-11, the area of $\triangle DEF$ is nine times the area of $\triangle LMN$.

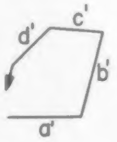
We now turn our attention from triangles to polygons with any number of sides.

Theorem 11-12 is concerned with perimeters and of course applies to triangles as well as to other polygons. Theorem 11-13 generalizes Theorem 11-11 to the case of polygons with n sides.

THEOREM 11-12. Every similarity between convex polygons with n sides has the property that the lengths of the n sides and the perimeter of one polygon are proportional to the lengths of the corresponding sides and the perimeter of the other polygon.

Proof: Let the lengths of the sides of one convex polygon be $a, b, c, \ldots$, and let the perimeter $a + b + c \ldots$ be p. Let the lengths of the corresponding sides of the other convex polygon be a', b', c', ..., and let the perimeter $a' + b' + c' + \ldots$ be p'. Let k be the proportionality constant for the similarity.





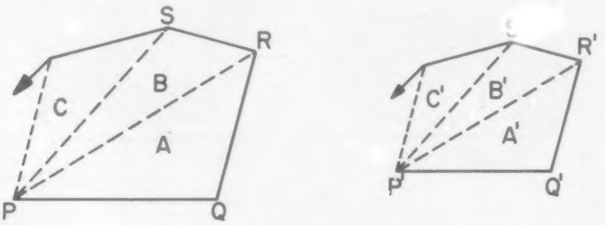
Then $a = ka'$, $b = kb'$, $c = kc'$, ... Hence

$$\begin{aligned} p &= a + b + c + \ldots \\ &= ka' + kb' + kc' + \ldots \\ &= k(a' + b' + c' + \ldots) \\ &= kp' . \end{aligned}$$

Thus a, b, ..., p are proportional to a', b', ..., p' with proportionality constant k.

THEOREM 11-13. Every similarity between convex polygons with n sides has the property that the areas of the polygonal-regions (consisting of the polygons and their interiors, respectively) are proportional to the squares of the lengths of any pair of corresponding sides.

Proof: We outline the proof; you are asked to supply the details as a problem in the next problem set. Let $PQR\ldots \longleftrightarrow P'Q'R'\ldots$ be a similarity between polygons with n sides. Let k be the proportionality constant. The diagonals from P cut the polygonal-region $PQR\ldots$ into triangular-regions; let the areas of these triangular-regions be A, B, ... In a like manner, let A', B', ... be the areas of the corresponding triangular-regions into which the diagonals from P' cut the polygonal-region $P'Q'R'\ldots$



Prove that $\triangle PQR \sim \triangle P'Q'R'$,

$\triangle PRS \sim \triangle P'R'S'$, etc.

By Theorem 11-11,

$$\frac{A}{A'} = k^2 , \ \frac{B}{B'} = k^2 , \ \frac{C}{C'} = k^2 , \text{ etc.}$$

Hence

$$A + B + C + \ldots = k^2A' + k^2B' + k^2C' + \ldots$$
$$= k^2(A' + B' + C' + \ldots) .$$

Thus

$$\frac{A + B + \ldots}{A' + B' + \ldots} = k^2 = \frac{(PQ)^2}{(P'Q')^2} = \frac{(QR)^2}{(Q'R')^2} = \ldots$$

Problem Set 11-7

1. The lengths of a pair of corresponding sides of two similar triangles are 4 and 5 . What is the ratio of the areas of the triangles?

2. The areas of two similar triangles are 64 and 100. What is the ratio of the lengths of corresponding sides? the ratio of corresponding altitudes? the ratio of perimeters?

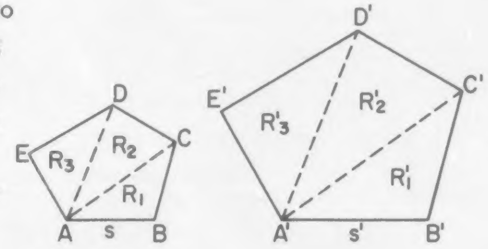
3. Two similar triangles are such that the area of the first triangle is 16 times the area of the other triangle. What is the ratio of the length of a side of the first triangle to the length of a corresponding side of the second?

4. The areas of two similar triangles are 64 and 100. If a side of the first measures 24, find the measure of the corresponding side of the second.

5. The altitude of an equilateral triangle is equal to the length of a side of a second equilateral triangle. What is the ratio of the lengths of corresponding sides? the ratio of the areas?

6. Cut a triangle into three polygonal-regions of equal area by drawing lines parallel to a base.

7. By hypothesis, we have two similar pentagons, ABCDE and A'B'C'D'E'. We are to prove that their areas are proportional to the squares of the lengths of any two corresponding sides.



Restatement: $\frac{\text{area } ABCDE}{\text{area } A'B'C'D'E'} = \frac{s^2}{s'^2}$.

(Draw diagonals from A and A' of the polygons.)

8. Problem 7 asks for the proof of Theorem 11-13 for the case of pentagons. Use the same ideas and give a proof of Theorem 11-13 for polygons with any number of sides.

9. The areas of two similar polygons are 144 and 256. If a side of the first measures 9, what is the measure of the corresponding side of the second?

10. The lengths of the corresponding diagonals of two similar polygons are 7 and 10 . What is the ratio of the areas? the perimeters?

11. Find the ratio of the perimeters of two regular octagons if the areas are 25 and 50 .

12. Prove that the area of a square having the diagonal of a given square as a side has twice the area of the given square.

13. Two similar polygons RSTUV and R'S'T'U'V' are such that $\angle R$ coincides with $\angle R'$. The coordinates of $R = R'$, of S , of S' are (2,2) , (2,11) , (2,8) , respectively. Find the ratio of the lengths of corresponding sides of the polygons; the ratio of perimeters; the ratio of areas.

14. The areas of two similar triangles are 144 and 81 . If a side of the former measures 6 , what is the length of the corresponding side of the latter?

15. In $\triangle ABC$, the point D is on side $\overline{AC}$, and AD is twice CD . Let the line $\overleftrightarrow{DE}$ parallel to $\overleftrightarrow{AB}$ intersect $\overleftrightarrow{BC}$ at E . Compare the areas of triangles ABC and DEC .

16. How long must a side of an equilateral triangle be in order that its area shall be twice that of an equilateral triangle whose side measures 10 ?

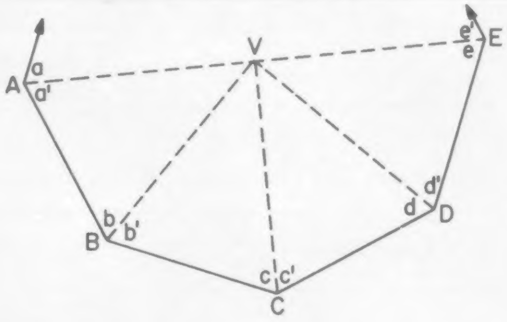
17. If similar triangles are drawn having, respectively, the side and the altitude of an equilateral triangle as corresponding sides, prove that the ratio of their areas is 4 to 3 .

18. Two pieces of wire of equal length are bent to form a square and an equilateral triangle respectively. What is the ratio of the areas of the two polygonal-regions bounded by the respective polygons?

11-8. Regular Polygons.

THEOREM 11-14. The bisectors of the interior angles of a regular convex polygon of n sides intersect at a point.

Proof: Given a regular convex polygon ABCDEF... with $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE} \cong \overline{EF}$, etc., and $\angle A \cong \angle B \cong \angle C \cong \angle D \cong \angle E \cong \angle F$, etc.



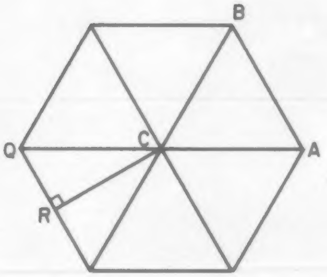
Let the bisectors of $\angle A$ and $\angle B$ intersect at point V. Then $\triangle AVB$ is isosceles, because $m\angle a' = \frac{1}{2}m\angle A = \frac{1}{2}m\angle B = m\angle b$ Thus $AV = BV$. Now $m\angle a' = m\angle b = m\angle b'$. Hence the correspondence $AVB \longleftrightarrow BVC$ between triangles is a congruence, by S.A.S. (Why?) Therefore $m\angle c = m\angle b = \frac{1}{2}m\angle B = \frac{1}{2}m\angle C$. That is, $\overrightarrow{CV}$ is the bisector of $\angle C$. In a like manner, we can prove that $\triangle BVC$ is isosceles, that the correspondence $BVC \longleftrightarrow CVD$ is a congruence between triangles, and that $\overrightarrow{DV}$ bisects $\angle D$. The same procedure shows that $\overrightarrow{EV}$ bisects $\angle E$, etc. In summary, all the bisectors meet at the point V.

DEFINITIONS. The center of a regular polygon is the point of intersection of the midrays of any two angles of the polygon.

Any triangle whose vertices are the center and two consecutive vertices of the polygon is called a central triangle of the regular polygon.

A radius of a regular polygon is any segment joining the center and a vertex of the polygon.

An apothem of a regular polygon is any segment which joins the center and a side of the polygon and is perpendicular to that side.



As an example, the center of the regular hexagon in the diagram is C . There are six central triangles, one of which is $\triangle ABC$. The segment $\overline{CQ}$ is a radius and the segment $\overline{CR}$ is an apothem of the regular hexagon.

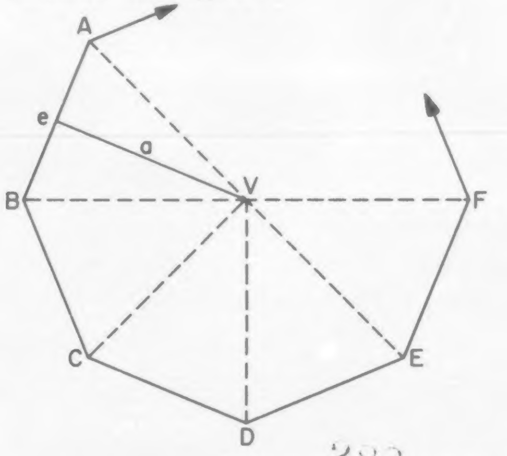
Theorem 11-14 tells us that the center of a regular polygon is the point of intersection of all the bisectors of angles of the polygon.

THEOREM 11-15. Every central triangle of a regular polygon is isosceles and is congruent to every other central triangle.

Proof: These statements, expressed now in the new language of "central triangle," were actually established in the proof of Theorem 11-14. Indeed, using the notation of that proof, we showed that each of the central triangles AVB , BVC , CVD , etc., is isosceles and that $\triangle AVB \cong \triangle BVC \cong \triangle CVD \cong \ldots$

THEOREM 11-16. The area of a regular polygon is one-half the product of the apothem and the perimeter of the polygon.

Proof: Let ABC... be a regular polygon with n sides. Let V be the center of the polygon, let a be the apothem, and let e be the length of one side of the polygon. The segments joining the center V and the vertices of the polygon determine n central triangles.



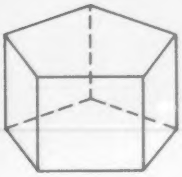
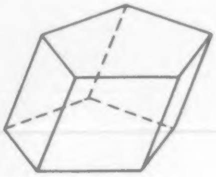
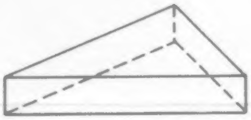
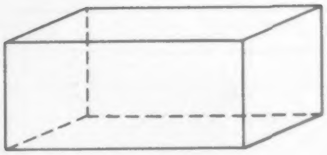
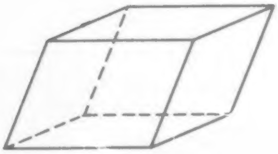
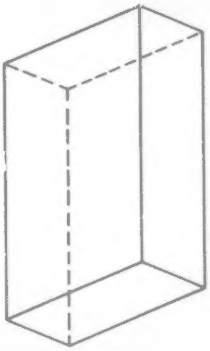
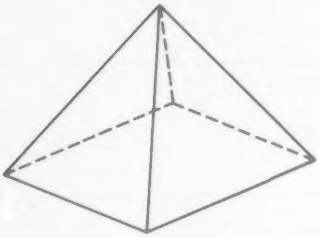
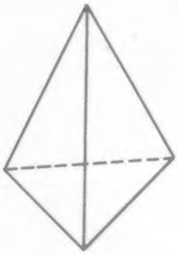
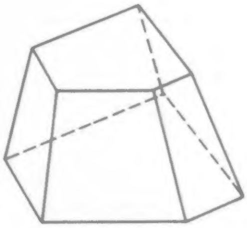
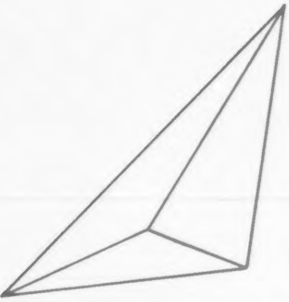
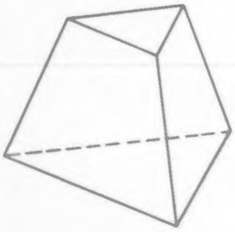
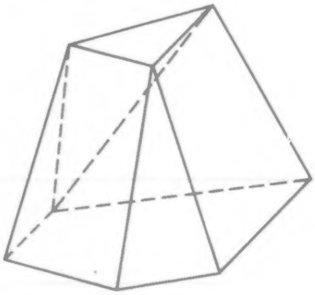
Each of these central triangles has base e and altitude a and hence area $\frac{1}{2}ea$. The area of the regular polygon is therefore $n(\frac{1}{2}ea) = \frac{1}{2}a(ne)$. Since ne is the perimeter of the polygon, the theorem is proved.

Problem Set 11-8

1. Does a perpendicular segment from the center of a regular polygon to a side bisect the side? Why?
2. The apothem of a regular hexagon is $10\sqrt{3}$. What is the length of each side of the hexagon?
3. The diagonal of a square has length $6\sqrt{2}$. What is the radius? the perimeter? the apothem? the area?
4. Given an equilateral triangle whose side measures s , find the radius and the apothem of the triangle in terms of s .
5. The perimeter of a regular hexagon is 12 . Find the apothem, the radius, the area.
6. The radius of a square is r . Find the apothem, the length of a side, the perimeter, and the area of the square all in terms of r .
7. The apothems of two equilateral triangles are 8 and 12 .
   (a) What is the ratio of the radii? of the lengths of their sides? of the perimeters? of the areas?
   (b) Find the area of the smaller triangle by two different methods.
8. (a) Each side of a regular hexagon is $8\sqrt{3}$. Find the area of the hexagon.
   (b) The apothem of a regular hexagon is 12 . What is the perimeter of the hexagon? the area?
   (c) Use another method to find the area of the hexagon in (b) .

11-9. Polyhedrons.

Pictures of various polyhedrons look like the following:

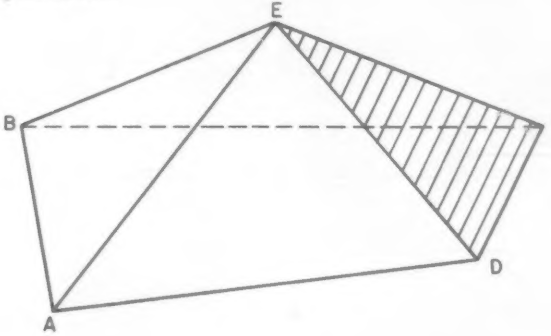













DEFINITIONS. A *polyhedron* is the union of a finite number of polygonal-regions, each of which consists of a convex polygon and its interior, such that (1) the interiors of any two of the polygonal-regions do not intersect and (2) every side of any of the polygons is also a side of exactly one other of the polygons.

Each vertex of any of these polygons is called a *vertex* of the polyhedron.

Each side of any of these polygons is called an *edge* of the polyhedron.

Each of the polygonal-regions is called a *face* of the polyhedron.



As an example, consider the polyhedron in the above diagram. It has five vertices. It has eight edges, two of which are $\overline{BC}$ and $\overline{AE}$. It has five faces, one of which is the shaded triangular-region CDE .

A polyhedron is named according to the number of faces which it contains. Since the number of sides of a polygon is the basis for naming a polygon, we expect some resemblance between the names of polygons and the names of polyhedrons. The following table shows this analogy.

Name of Polygon	Number of Sides	Name of Polyhedron	Number of Faces
Triangle	3	(No polyhedron has three faces.)	
Quadrilateral	4	Tetrahedron	4
Pentagon	5	Pentahedron	5
Hexagon	6	Hexahedron	6
Heptagon	7	Heptahedron	7
Octagon	8	Octahedron	8
Nonagon	9	Nonahedron	9
Decagon	10	Decahedron	10
Dodecagon	12	Dodecahedron	12
20-gon	20	Icosahedron	20

Prisms, pyramids, and frustums of pyramids are examples of special kinds of polyhedrons. Other examples are the so-called regular polyhedrons.

DEFINITIONS. Any non-empty intersection of a polyhedron and a plane is called a *section* of the polyhedron.

A polyhedron is a *convex polyhedron* if and only if every section of it which contains at least three non-collinear points is either a convex polygon or a face of the polyhedron.

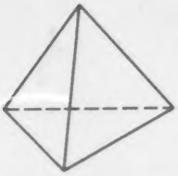
A *regular polyhedron* is a convex polyhedron such that:

(1) each face is the union of a regular polygon and its interior;

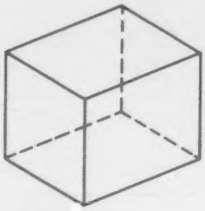
(2) all these regular polygons have the same number of sides; and

(3) all vertices of the polyhedron belong to the same number of faces.

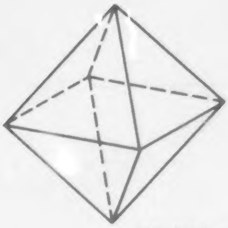
It is interesting to note that there are only five types of regular polyhedrons: the regular tetrahedron, the regular hexahedron (also called the cube), the regular octahedron, the regular dodecahedron, the regular icosahedron. This fact will be discussed again later in the chapter. Pictures of these five types of polyhedrons are shown below.



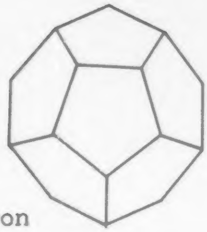
Tetrahedron



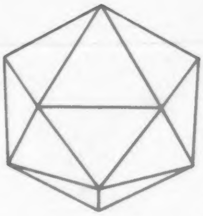
Hexahedron
or cube



Octahedron

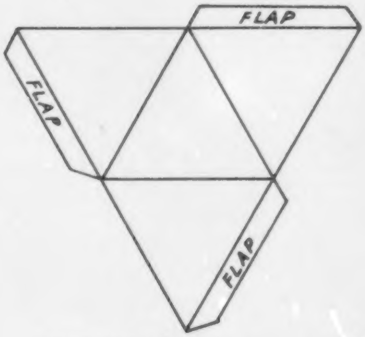


Dodecahedron

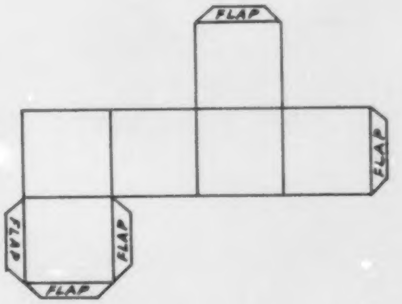


Icosahedron

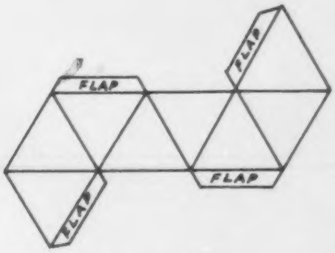
Models of regular polyhedrons are not difficult to make, and they are very helpful in studying the properties of the regular polyhedrons. The plans for making these models are given below. They should be constructed from stiff paper, using dimensions that are at least five times as large as the dimensions of the pattern.


FLAP
FLAP
FLAP

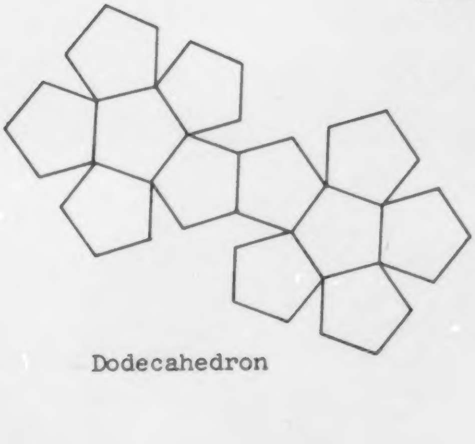
Tetrahedron


FLAP
FLAP
FLAP
FLAP
FLAP

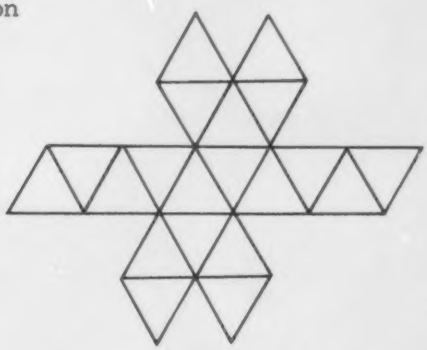
Hexahedron


FLAP
FLAP
FLAP
FLAP

Octahedron



Dodecahedron



Icosahedron

Problem Set 11-9

1. Make a table similar to the following and fill in the blanks for the indicated regular polyhedrons.

Regular Polyhedron	Boundary of Face	Number of Faces	Number of Edges	Number of Vertices	Number of Faces (or Edges) at a Vertex
Tetrahedron					
Octahedron					
Icosahedron					
Hexahedron					
Dodecahedron					

2. From the preceding table, verify the formula $f - e + v = 2$ where f is the number of faces of the regular polyhedron, e is the number of edges, and v is the number of vertices. Do you think the formula is also true for polyhedrons which are not regular polyhedrons?

3. Explain why there is no polyhedron with three faces.

If you would like to know more about the relations that exist among regular polyhedrons, or if you are interested in constructing models that use regular polyhedrons as a basis for their construction, the following books will be of interest to you.

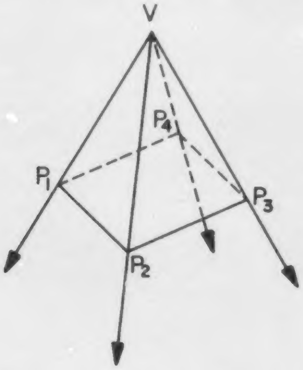
Steinhaus, Mathematical Snapshots

Cundy and Rollett, Mathematical Models

11-10. Polyhedral Angles.

In Chapters 4 and 9 we studied plane angles and dihedral angles. In this section we introduce another type of angle known as the polyhedral angle. We also study some important properties of polyhedral angles.

A picture of a polyhedral angle is the following:



This polyhedral angle is determined by the convex quadrilateral $P_1P_2P_3P_4$ and the point V not in the plane of the quadrilateral. The rays $\overrightarrow{VP_1}$, $\overrightarrow{VP_2}$, $\overrightarrow{VP_3}$, $\overrightarrow{VP_4}$, are edges of the polyhedral angle. Each of four angles at V, namely $\angle P_1VP_2$, $\angle P_2VP_3$, $\angle P_3VP_4$, $\angle P_4VP_1$ is a face angle of the polyhedral angle. A face of the polyhedral angle is the union of a face angle and its interior; for example, in the plane VP_1P_4, the union of $\angle P_4VP_1$ and its interior is a face. The polyhedral angle itself is the set of all points belonging to any of the faces. This illustration leads us to the following definitions.

> DEFINITIONS. Let a convex polygon and a point V not in the plane containing the polygon be given; the union of all the concurrent rays which have endpoint V and which contain a point of the polygon is called a polyhedral angle.
>
> The point V is the vertex of the polyhedral angle.
>
> Each ray with endpoint V and containing a vertex of the polygon is an edge of the polyhedral angle.

An angle with vertex V and containing two consecutive vertices of the polygon is a face angle of the polyhedral angle.

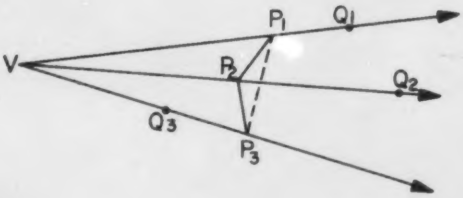
A face of a polyhedral angle is the union of a face angle and its interior.

A polyhedral angle of three faces is called a trihedral angle.

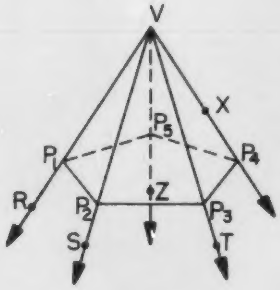
Notation. If a polyhedral angle is determined by the convex polygon $P_1P_2\ldots P_n$ and the vertex V, if Q_1 is an interior point of $\overrightarrow{VP_1}$, if Q_2 is an interior point of $\overrightarrow{VP_2}, \ldots,$ and if Q_n is an interior point of $\overrightarrow{VP_n}$, then the polyhedral angle is denoted by the symbol $\angle V - Q_1Q_2\ldots Q_n$.

In particular, the polyhedral angle may be denoted by $\angle V - P_1P_2\ldots P_n$.

Other pictures of polyhedral angles are the following.



$\angle V - Q_1Q_2Q_3$



$\angle V$ - RSTXZ

Exploratory Problems

1. How many trihedral angles are formed by the walls, floor, ceiling of your classroom? What do you think is the measure of each of their face angles?

2. Can you make a model of a trihedral angle with exactly one of the face angles a right angle? With exactly two of the face angles as right angles? With every face angle a right angle? Is it possible to make a model of a polyhedral angle with four faces such that each of the face angles is a right angle? Explain.

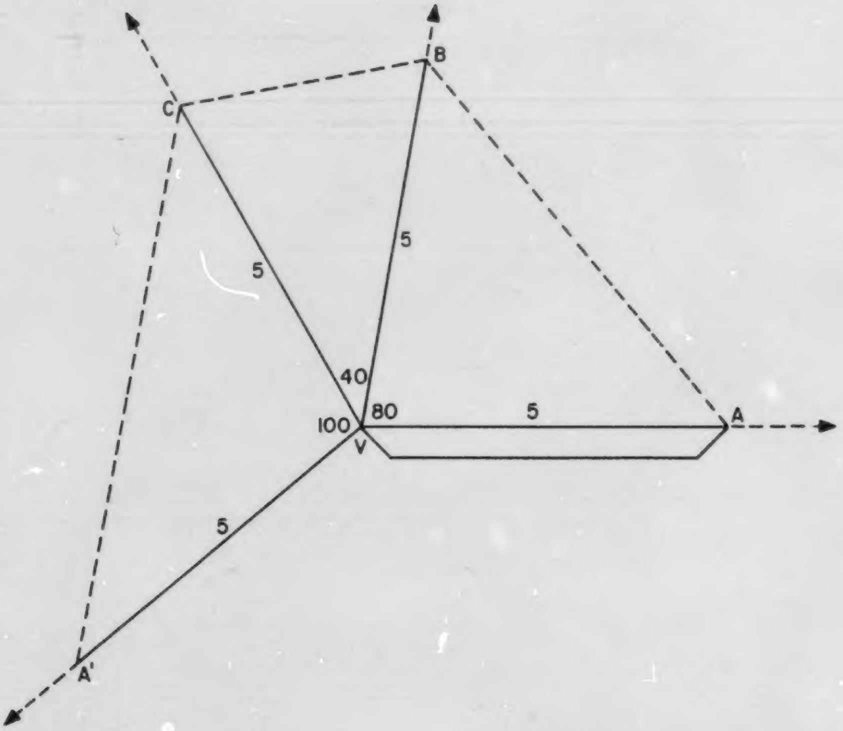
3. Make a model of a polyhedral angle with five faces so that each face angle measures 60 . Can you make a model of a polyhedral angle with six faces if each face angle measures 60 ? Explain.

4. Do you think it is possible for a polyhedral angle to have four face angles whose respective measures are 50 , 120 , 90 , 100 ? Explain.

5. Complete the following statement: The sum of the measures of the face angles of a polyhedral angle is ____________ ____________________.

6. Construct a model of a trihedral angle, say $\angle V - ABC$, such that the measures of the face angles $\angle AVB$, $\angle BVC$, $\angle CVA$ are 80 , 40 , 100 , respectively. (The pattern of such a model is given in the diagram below. The suggested distances are measured in inches. As you complete the model by bringing A and A' together, keep the rays "pointing downward from" the vertex V and keep face AVB toward your right.)

   Compare your model with those of your classmates. Do you think that all the trihedral angles represented by these models are congruent to each other?



7. Construct, as in Problem 6, a model of another trihedral angle, say $\angle W - DEF$, where the measures of $\angle DWE$, $\angle EWF$, $\angle FWD$ are 40, 80, 100, respectively. Compare your model with those of your classmates. Do all these trihedral angles appear to be congruent?

8. Does the trihedral angle whose model you constructed in Problem 7 appear to be congruent to the trihedral angle whose model you constructed in Problem 6 ?

9. The models which you constructed in Problems 6 and 7 give an example of a pair of "symmetric" trihedral angles. What do you think is meant by saying that two trihedral angles are symmetric to each other?

10. Draw pictures of a pair of vertical trihedral angles. Do they appear to be congruent? Do they appear to be symmetric? (You should be able to guess the meaning of "vertical" trihedral angles by analogy with vertical angles.)

11. Try and make models of trihedral angles with face angles measuring:

    (a) 40 , 50 , 100 , respectively;

    (b) 40 , 50 , 90 , respectively;

    (c) 40 , 50 , 80 , respectively.

12. Explain the result of Problem 11.

13. Complete the following sentence: The sum of the measures of two face angles of a trihedral angle is _____ _______.

The preceding exploratory problems lead us to the following two theorems, whose proofs we omit.

THEOREM 11-17. The sum of the measures of any two face angles of a trihedral angle is greater than the measure of the third face angle.

THEOREM 11-18. The sum of the measures of all the face angles of any polyhedral angle is less than 360 .

As an application of the preceding two theorems, consider the following situation. Suppose that the measures of two of the face angles of a trihedral angle are known to be 75 and 115 . We ask what information can be deduced about the measure of the third face angle of this polyhedral angle. Let the measure of the third face angle be denoted by x .

(1) By Theorem 11-17, we find that:

$$x + 75 > 115 ,$$

$$x + 115 > 75 , \text{ and}$$

$$75 + 115 > x .$$

The first of these inequalities tells us that

$$x > 40 \qquad \text{(Why?)} ;$$

since x is positive, the second of the inequalities gives us no new information; and the third of the inequalities, namely $x < 190$, also does not provide any new information about the number x (why?) .

(2) By Theorem 11-18, we find that

$$x + 75 + 115 < 360 .$$

Hence

$$x < 170 .$$

(3) Since Part (1) tells us that $x > 40$ and Part (2) tells us that $x < 170$, we finally conclude that

$$40 < x < 170 .$$

Problem Set 11-10

1. In each of the following, the measures of two of the face angles of a trihedral angle are given. Find two numbers such that the measure of the third face angle is between them, in accordance with the information provided by Theorems 11-17 and 11-18.

(a) 80 , 105
(b) 100 , 125
(c) 60 , 135
(d) 145 , 175
(e) 50 , 135
(f) 80 , 95

2. True - False statements. Write + if the statement is true; write 0 if the statement is false.

(a) Each of the three face angles of a trihedral angle can be obtuse.
(b) A polyhedral angle can have four face angles that are right angles.
(c) The measure of the face angles of a polyhedral angle with four faces can be 50 , 65 , 100 , and 110 .

(d) The measures of the face angles of a trihedral angle can be 140 , 130 , and 120 .

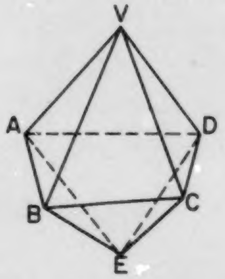
(e) If the measures of two face angles of a trihedral angle are 100 and 120 , the measure of the third face angle is less than 20 .

(f) If each face angle of a polyhedral angle measures 60 , the polyhedral angle must be a trihedral angle.

(g) If the measure of each face angle of a polyhedral angle is 90 , the polyhedral angle must have four faces.

(h) If a plane is perpendicular to one edge of a polyhedral angle, it is perpendicular to two faces of the polyhedral angle.

Corresponding to each vertex V of a convex polyhedron, there is a polyhedral angle, whose vertex is V and whose edges are the rays containing those edges of the polyhedron that have an endpoint at V. In the illustrative diagram at the right, the polyhedral angle associated with vertex V of the polyhedron VABCDE is $\angle V - ABCD$. The faces of the polyhedral angle with vertex V contain the respective faces of the polyhedron that contain the point V .



In the preceding section we described the so-called regular polyhedrons. By pictures and models we found five types of regular polyhedrons. The number of respective faces is 4 , 6 , 8 , 12 , 20 .

The length of an edge of a cube (regular hexahedron) may be any positive number. So, although cubes can occur in any "size," they all have the same "shape," in other words, they are similar to one another. In a like manner, regular tetrahedrons of different "sizes" are nevertheless similar to each other. In general, regular polyhedrons of any of the five types we have studied are similar to one another. The remarkable fact is that these five types are the only types of regular polyhedrons that exist. We formulate this as the next theorem, whose proof we merely sketch.

THEOREM 11-19. There are no more than five types of regular polyhedrons.

Outline of proof:

(1) A polyhedral angle has at least three face angles.

(2) The sum of the measures of the face angles of a polyhedral angle is less than 360 .

(3) The face angles of the polyhedral angle corresponding to each vertex of a regular polyhedron have the same measure.

(4) Therefore the measure of each face angle must be less than 120 .

(5) The measure of each angle of a regular polygon with 6 or more sides is at least 120 .

(6) Hence every face of a regular polyhedron has less than 6 edges; in other words, a face of a regular polyhedron is a polygonal-region whose boundary has either 3 or 4 or 5 sides.

(7) Suppose that each face has 3 edges. Then,

  (a) each face angle has measure 60 ;
  (b) each polyhedral angle can have 3 or 4 or 5 faces, by parts (1) and (2);
  (c) no more than three types of regular polyhedrons have faces which are triangular-regions.

(8) Suppose that each face has 4 edges. Then,

  (a) each face angle has measure 90 ;
  (b) each polyhedral angle has exactly 3 faces, by parts (1) and (2);
  (c) no more than one type of regular polyhedron has faces which are squares and their interiors.

(9) Suppose that each face has 5 edges. Then,

   (a) each face angle has measure 108 ;
   (b) each polyhedral angle has exactly 3 faces;
   (c) no more than one type of regular polyhedron has faces which are pentagons and their interiors.

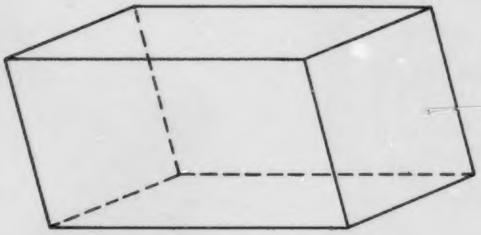
(10) In summary, there are no more than five types of regular polyhedrons.

11-11. Prisms.

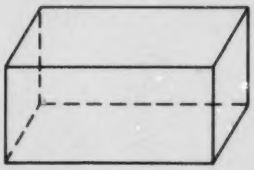
We now study another type of polyhedron, namely the prism.

DEFINITION. A prism is a polyhedron such that two of its faces (called bases) have boundaries which are congruent polygons in parallel planes and each of the remaining faces has a boundary which is a parallelogram with two sides in the parallel planes.

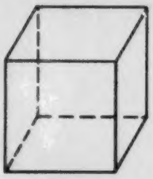
Prisms are classified according to their bases: A prism each of whose bases is a triangular-region is called a triangular prism; a prism each of whose bases is a rectangular-region is called a rectangular prism; and so on. Of particular importance among the prisms each of whose bases has a quadrilateral as a boundary are the following:



Parallelepiped



Rectangular Parallelepiped



Cube

DEFINITIONS. A parallelepiped is a prism such that the boundary of each of its bases is a parallelogram.

A rectangular parallelepiped is a parallelepiped such that the boundary of each of its faces is a rectangle.

A cube is a parallelepiped such that the boundary of each of its faces is a square.

Notice that each parallelepiped is a polyhedron with six faces, that is, is a hexahedron. In particular, the cube is the regular hexahedron.

An ordinary box is a model of a rectangular parallelepiped. A prism such as a rectangular parallelepiped has three pairs of faces, each of which may be considered as a pair of bases. Is this also true of a parallelepiped which is not rectangular? Why? By contrast, only one pair of faces of a triangular prism may be considered as the two bases. Why?

DEFINITIONS. With reference to a selected pair of bases of a prism, we define the following:

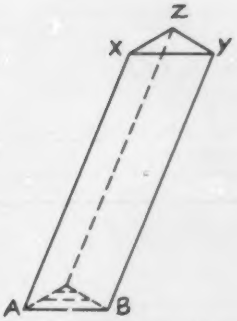
any one of the remaining faces is called a lateral face of the prism;

the union of the lateral faces is called the lateral surface of the prism (sometimes known as a prismatic surface);

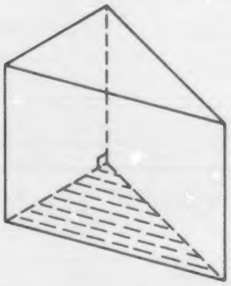
any edge which is the intersection of two lateral faces is called a lateral edge of the prism;

the prism is said to be a right prism if and only if a lateral edge is perpendicular to a base.

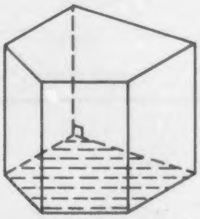
The left-hand figure below is a picture of a prism that is not a right prism. Face ABYX is a lateral face and $\overline{CZ}$ is a lateral edge of the triangular prism. Each of the other two diagrams below is a picture of a right prism.



Triangular prism



Right triangular prism



Right pentagonal prism

Problem Set 11-11a

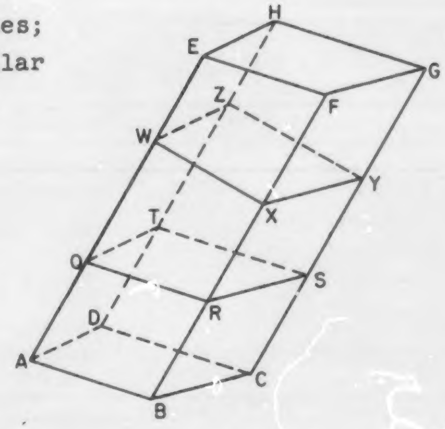
1. Explain why all the lateral edges of a prism are parallel to one another.
2. Prove that in a right prism every lateral edge is perpendicular to each base.

DEFINITIONS. With reference to a selected pair of bases of a prism, we define the following:

a cross-section of the prism is any non-empty intersection of the prism and a plane which is parallel to, and distinct from, the planes containing the bases;

a right-section is any intersection of the prism and a plane which is perpendicular to, and intersects the interior of, every lateral edge.

In the diagram at the right, plane ABCD , plane EFGH , and plane QRST are parallel planes; and plane WXYZ is perpendicular to $\overleftrightarrow{AE}$. Quadrilateral QRST is a cross-section of the prism and quadrilateral WXYZ is a right-section of the prism.



DEFINITIONS. With reference to a selected pair of bases of a prism, we define the following:

any segment whose endpoints lie in the two parallel planes containing the bases and which is perpendicular to these planes is an altitude of the prism;

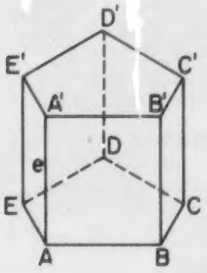
the sum of the areas of all the lateral faces of the prism is the lateral area of the prism;

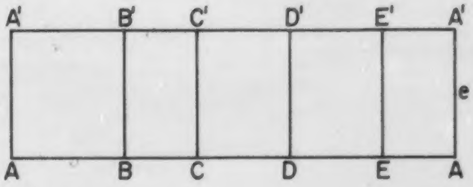
the sum of the areas of all the faces of the prism is the total area of the prism.

A method of computing the lateral area of a prism is to find the area of each of the lateral faces and then to add their areas. The following experiments help you to recognize a simpler method for finding the lateral area of a prism.

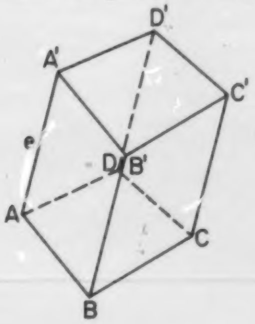
Experiments

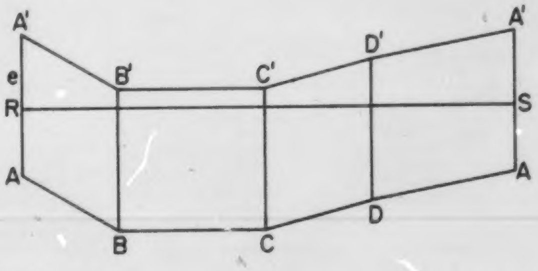
1. Cut along one of the edges of a right prismatic surface. Note that this surface can be flattened into a rectangle as shown in the next figure. The base of the rectangle is the __________ of the ________ of the prism, and the altitude of the rectangle is the ___________ of the prism. Therefore, the lateral area of the prism is the product of ____________ and ______________.





2. Cut along the lateral edge of a prismatic surface that is not a right prismatic surface. Flatten this prismatic surface into a plane surface as shown in the following figure.



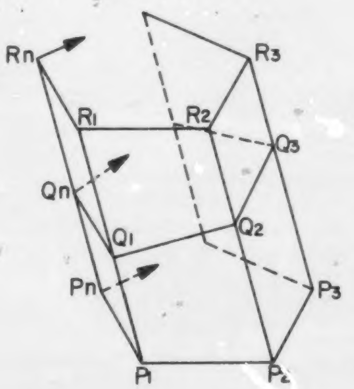


Draw a line in the plane perpendicular to one of the edges of the flattened surface as shown. Does the length of $\overline{RS}$ equal the sum of the altitudes of the parallelograms which are the lateral faces of the prism? Why does the lateral area of the prism equal the product of the lengths of $\overline{RS}$ and a lateral edge of the prism?

Change the flattened figure back to the original prismatic surface. The length of $\overline{RS}$ is the same as the perimeter of a ______________________ of the prism.

THEOREM 11-20. The lateral area of a prism is equal to the product of the length of a lateral edge and the perimeter of a right-section.

Proof: We are given a prism with bases $P_1P_2...P_n$ and $R_1R_2...R_n$ and right section $Q_1Q_2...Q_n$. Let L be the lateral area of the prism, e the length of a lateral edge, and p the perimeter of the given right section. We are required to prove that $L = ep$.



Statements	Reasons
1. $P_1R_1 = P_2R_2 = P_3R_3 = \ldots = P_nR_n = e$.	1. Why?
2. $\overline{Q_nQ_1} \perp \overline{P_1R_1}$; $\overline{Q_1Q_2} \perp \overline{P_2R_2}$, etc.	2. Why?
3. Area of $P_1R_1R_2P_2$ is $e(Q_1Q_2)$, Area of $P_2R_2R_3P_3$ is $e(Q_2Q_3)$, Area of $P_3R_3R_4P_4$ is $e(Q_3Q_4)$, etc.	3. Why?
4. Sum of the areas of n parallelograms is $e(Q_1Q_2 + Q_2Q_3 + Q_3Q_4 + \ldots + Q_{n-1}Q_n + Q_nQ_1)$.	4. Why?
5. Therefore, $L = ep$.	5. Why?

Corollary 11-20-1. The lateral area of a right prism is the product of the length of a lateral edge and the perimeter of a base.

Problem Set 11-11b

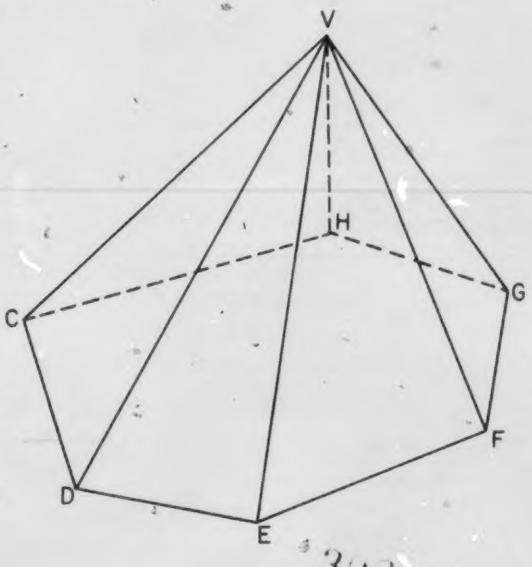
1. Supply the reasons for the statements in the proof of Theorem 11-20.
2. Prove Corollary 11-20-1.
3. Find the area of the lateral surface of a right prism whose altitude is 10 if the sides of the pentagonal base measure 3 , 4 , 5 , 7 , 2 , respectively.
4. Find the total area of a right triangular prism if the base is an equilateral triangle 8 inches on a side and the altitude of the prism is 10 inches.
5. If the sides of a cross-section of a right triangular prism measure 3 , 6 , and $3\sqrt{3}$, then any other cross section will be a triangle whose sides measure _____ , _____ , and _____ , and whose angles measure _____ , _____ , _____ , and whose area is _____ .
6. The length of a lateral edge of a right prism is 10 and its lateral area is 52 . What is the perimeter of the base of the prism?
7. The apothem of the base of a right hexagonal prism is $8\sqrt{3}$. The altitude of the prism is 20 . Find the lateral area of the prism.
8. At one of the vertices of a certain square prism, the associated polyhedral angle has face angles which measure 90 , 90 , 30 , respectively. Each lateral edge of the prism is 20 inches long, and the perimeter of the base is 48 inches. Find the total area of the prism.
9. Prove by the use of coordinates that the diagonals of a rectangular parallelepiped have equal length.
10. Prove by the use of coordinates that the diagonals of a rectangular parallelepiped bisect each other.

11-12. Pyramids.

Pyramids resemble prisms in several respects. Many of the terms such as lateral face, lateral edge, cross-section, and lateral area are the same, and we shall use them without formal definition.

> DEFINITIONS. A pyramid is a convex polyhedron which, except for the interior of one of its faces, is contained in a polyhedral angle.
>
> The vertex of the polyhedral angle is called the vertex of the pyramid.
>
> The face of the pyramid which is not contained in the polyhedral angle is called the base of the pyramid.
>
> The segment which joins the vertex and the plane containing the base and is perpendicular to that plane is called the altitude of the pyramid.

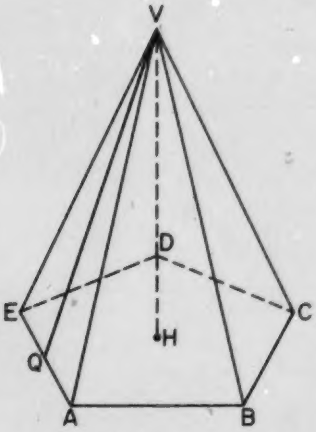
In the diagram below, V is the vertex of the pyramid; the polygonal-region CDEFGH is the base of the pyramid; with the exception of the interior of face CDEFGH, the pyramid is contained in $\angle V$ - CDEFGH .



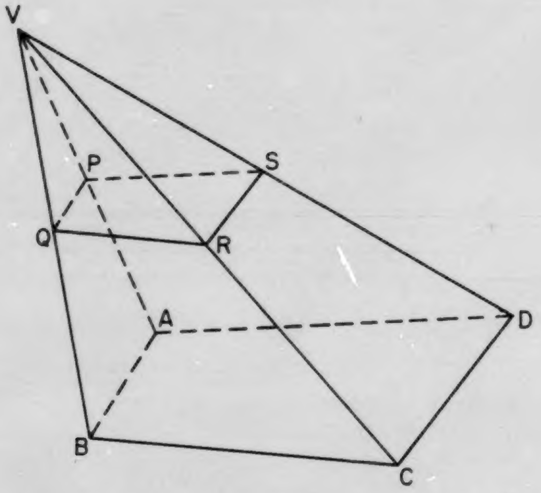
DEFINITIONS. A pyramid is a regular pyramid if and only if the boundary of its base is a regular polygon and the center of the base is an endpoint of the altitude of the pyramid.

The slant height of a regular pyramid is the distance between the vertex of the pyramid and an edge in the base of the pyramid.

In the following diagram, the altitude VQ of the isosceles triangle EVA is the slant height of the regular pentagonal pyramid.



Associated with the set of all pyramids is another important class of polyhedrons. If we imagine "cutting off the top" of a pyramid, the remaining figure suggests a frustum of the pyramid. In the diagram below, the polyhedron with vertices A , B , C , D , P , Q , R , S is a frustum of the pyramid whose vertex is V and whose base is the polygonal-region ABCD .



DEFINITION. Given a pyramid, a *frustum of the pyramid* is a polyhedron such that:

(1) one of its faces is the base of the pyramid;

(2) another of its faces is in a plane parallel to the plane containing the base of the pyramid, and

(3) each of its other faces is contained in the pyramid.

The proof of the following theorem is left as a problem.

THEOREM 11-21. Let a triangular pyramid be given.

(a) Every cross-section of the pyramid is a triangle similar to the boundary of the base.

(b) If the distance from the vertex of the pyramid to the plane containing the cross-section is k and if the altitude of the pyramid is h, then the area of the cross-section and the area of the base are proportional to the numbers k^2 and h^2.

Problem Set 11-12

1. Prove that the boundaries of the lateral faces of a regular pyramid are isosceles triangles which are congruent to one another.

2. Prove that the lateral area of a regular pyramid is given by the formula $A = \frac{1}{2}ap$, in which p is the perimeter of the base and a is the slant height.

3. If p is the perimeter of the base of a regular pyramid and a is the slant height, find the lateral area of the pyramid, in each of the following cases.

   (a) $p = 18$, $a = 5\frac{1}{2}$.

   (b) $p = 2\frac{1}{2}$ (yards) , $a = 2\frac{1}{2}$ (feet) .

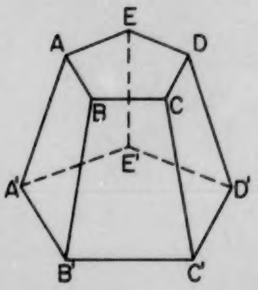
4. Find the area of the lateral surface of a regular square pyramid if each side of the base is 8 inches long and the slant height of the pyramid is 5 inches long.

5. What is the slant height of a regular pyramid if the area of its lateral surface is 80 and the perimeter of its base is 20 ?

6. Find the altitude of a regular square pyramid with a lateral edge measuring 25 and a diagonal of the base measuring 14 .

7. Fill the blank: The boundary of each lateral face of a frustum of a pyramid is a ____________ .

8. Derive a formula for the lateral area of a regular frustum (such as that shown in the diagram) if p is the perimeter of the lower base, p' is the perimeter of the upper base, and a is the altitude of a lateral face of the frustum.



9. The bases of a frustum of a regular pyramid are squares 8 inches and 6 inches on a side. The altitude of a lateral face of the frustum is 4 inches. Find the lateral area and the total area of the frustrum.

10. In a frustum of a regular square pyramid, an edge of the lower base measures 14 and an edge of the upper base measures 8 . If the lateral area of the frustum is 220 , find the altitude of a lateral face of the frustum.

11. In a pyramid with vertex V , rectangular base ABCD and altitude $\overline{VO}$ (as in the diagram), let EFGH be a cross-section similar to the rectangle ABCD such that the proportionality constant is $\frac{2}{3}$.

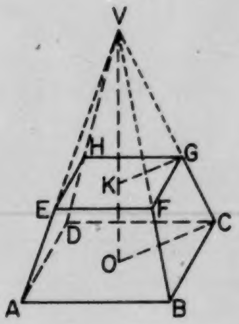
   (a) $\triangle FVG \sim \triangle BVC$. Why?

   (b) $\overline{VO} \perp \overline{OC}$. Why?

   (c) $\triangle KVG \sim \triangle OVC$. Why?

   (d) What is the ratio of VK to VO ?

   (e) Suppose the perimeter of the rectangle ABCD is 36 . What is the perimeter of the rectangle EFGH ?



(f) Suppose the altitude of a lateral face of the pyramid is 18 . What is the altitude of a corresponding lateral face of the frustum?

(g) What is the area of the lateral surface of the pyramid?

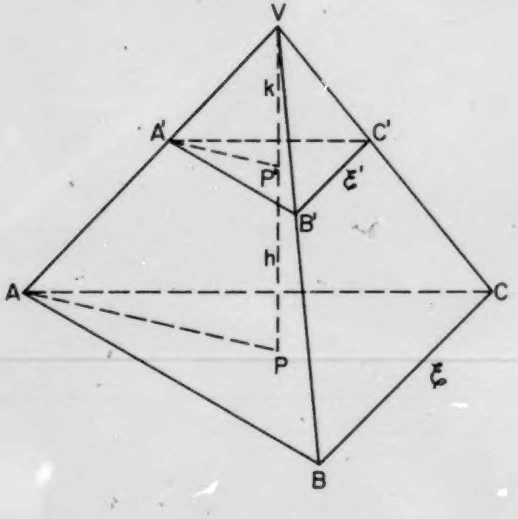
(h) What is the area of the lateral surface of the frustum?

(i) What is the ratio of the lateral area of the pyramid to the lateral area of the frustum? Explain.

12. Prove Theorem 11-21.

Hint: Let $\triangle ABC$ be in plane $\mathcal{E}$ and point V a distance h from $\mathcal{E}$. Let plane $\mathcal{E}'$, parallel to $\mathcal{E}$ and at distance k from V , intersect $\overline{VA}$, $\overline{VB}$, $\overline{VC}$ in A' , B' , C' , respectively. Then show that $\triangle A'B'C' \sim \triangle ABC$ and that

$$\frac{\text{area of } \triangle A'B'C'}{\text{area of } \triangle ABC} = \left(\frac{k}{h}\right)^2 .$$



311

13. A point of light is 6 feet from a wall. A piece of cardboard containing 24 square inches of surface area is held between the light and the wall 3 feet from the wall and parallel to it. Find the area of the shadow of the cardboard on the wall.

14. A point of light is 10 feet from a wall. How far from the wall, but parallel to it, should a piece of paper be held so as to cast a shadow four times the area of the paper?

11-13. Summary.

There are four major types of measures in our geometry. In Chapter 3, we discussed the measure of the distance between two points, or equivalently, the length of the segment joining the points. This is the basis for measuring a figure in one-dimensional geometry. In the present chapter, we treated a measure of convex polygons, or more generally, the area of any polygonal-region. This is the basis for measuring a figure in two-dimensional geometry. The next stage in this development would be (if we only had time!) to examine a measure of a convex polyhedron, or more generally, the volume of any "polyhedral-region." This would be the basis for measuring a figure in three-dimensional geometry. In Chapter 4, we considered the measure of an angle, or if you like, a measure of the "angular distance" between two concurrent rays.

There are extensions of measurement. The measure of a dihedral angle is defined in terms of the measure of an angle. In the next chapter the measure of an angle will permit us to measure an arc of a circle. Also in the next chapter the area of a polygonal-region will permit us to describe the areas of circular-regions and of other two-dimensional regions associated with circles. The volume of a polyhedral-region would permit us to discuss the volumes of spherical regions and other regions in space.

These types of measure have certain properties in common. (1) A measure is a real number that is not negative. (2) A measure depends upon a chosen unit, and the measures of the same geometric figure relative to different units are related in a simple manner. (3) The measures of two congruent figures are the same. (4) The measure of a "whole" is the sum of the measures of its nonoverlapping "parts"; that is, the length of a segment is the sum of the measures of any segments such that the interiors of any two of them do not intersect and their union is the given segment; the area of a polygonal-region is the sum of the measures of any polygonal-regions such that the interiors of any two of them do not intersect and their union is the given polygonal-region; a similar remark would apply to the volume of a polyhedral-region; the measure of an angle is the sum of the measures of any angles such that the interiors of any two of them do not intersect and the union of the angles and their interiors is the same as the union of the given angle and its interior.

There are connections among the various types of measure. The area of a two-dimensional region may be related to a product of two lengths. The volume of a three-dimensional region may be related to a product of three lengths, or to a product of a length and an area. These relationships are the familiar formulas for calculating areas and volumes. Their practical importance depends heavily on the fact that in the physical world it is often less convenient to measure an area or a volume directly than to compute it from data obtained by measuring appropriate distances and perhaps angles.

A similarity between two geometric figures either directly or indirectly prescribes many corresponding parts: sides, angles, diagonals, altitudes, medians, faces, bases, and so forth. In a similarity, the lengths of all segments, the square roots of the areas of all polygonal-regions, and the cube roots of the volumes of all polyhedral-regions in one geometric figure are proportional to the corresponding numbers for the other geometric figure. For the special case in which the constant of proportionality is one, the similarity is a congruence.

Review Problems

1. What is the measure of each interior angle of a regular polygon of fifteen sides? What is the measure of each exterior angle?

2. If an exterior angle of a regular polygon has measure 10, how many sides has the polygon?

3. The sum of the measures of the angles of a ten-sided polygon is 1440. Is the sum of the measures of the angles of a twenty-sided polygon twice as much? Verify your answer.

4. The hypotenuse of a 30, 60, 90 triangle is 16 and the shorter leg of a second 30, 60, 90 triangle is 13. What is the ratio of their areas?

5. Two face angles of a trihedral angle measure 56 and 100. Between what numbers must the measure of the third face angle of the trihedral angle be?

6. Which of the following triples of numbers can be the measures of the three face angles of a trihedral angle?

   (a) 45, 45, 90. (c) 140, 171, 70.

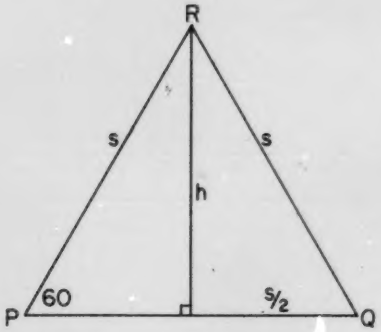
   (b) 60, 60, 60. (d) 150, 118, 130.

7. In choosing tile for a floor covering, would congruent regular hexagonal tiles give a complete coverage? What other regular polygons would fit together? How do you know?

8. By hypothesis we have an equilateral triangle with side measuring s, altitude h, and area A.

   (a) Show that $h = \frac{s}{2}\sqrt{3}$.

   (b) Show that $A = \frac{\sqrt{3}}{4}s^2$.



9. Find the area of an equilateral triangle having the length of a side equal to:

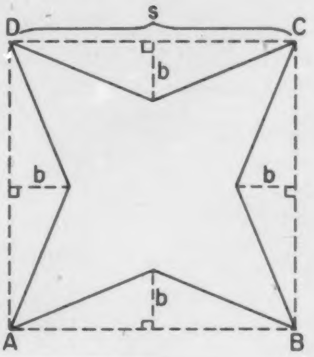
(a) 2	(c) $\sqrt{3}$
(b) 8	(d) 7

10. The area of an equilateral triangle is $9\sqrt{3}$. Find the length of its side and its altitude.

11. The altitude of an equilateral triangle is $8\sqrt{3}$. Determine the length of its side and its area.

12. A square whose area is 81 has its perimeter equal to the perimeter of an equilateral triangle. Find the area of the equilateral triangle.

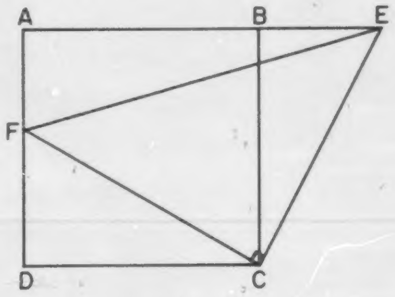
13. If ABCD is a square, find, in terms of s and b , the area of the star pictured here. The segments forming the boundary of the star are congruent.



14. Prove that the area of an isosceles right triangle is equal to one-fourth the area of a square having the hypotenuse of the triangle as a side.

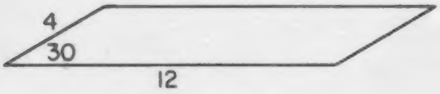
15. Given: Square ABCD with points E and F as shown, so that $\overline{EC} \perp \overline{FC}$. Area of ABCD is 256 . Area of $\triangle CEF$ is 200 .

Find BE .



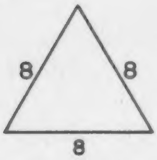
16. Find the area of each of the following polygonal-regions, if possible:

(a)



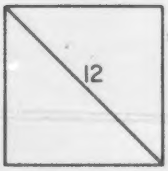
Parallelogram

(b)



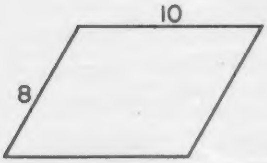
Triangle

(c)



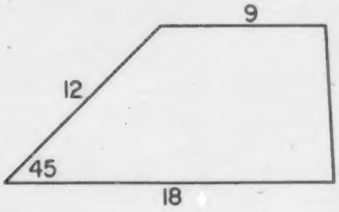
Square

(d)



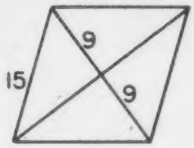
Parallelogram

(e)



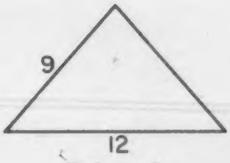
Trapezoid

(f)



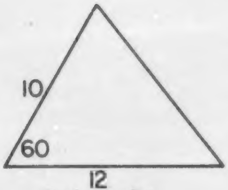
Rhombus

(g)



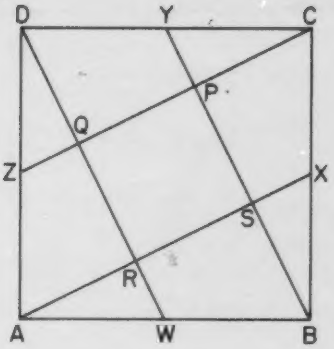
Triangle

(h)



Triangle

17. If W , X , Y and Z are midpoints of sides of square ABCD , as shown in the figure, compare the area of ABCD with the area of square RSPQ .

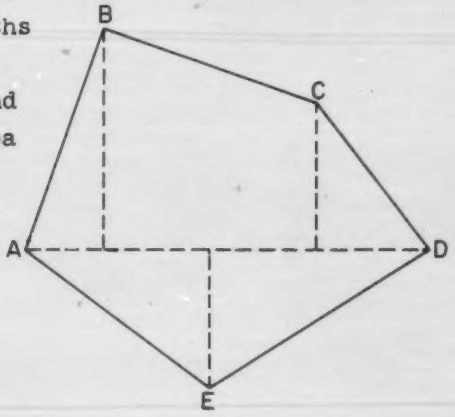


18. The area of a convex quadrilateral is 126 and the length of one diagonal is 21 . If the diagonals are perpendicular, find the length of the other diagonal.

19. The diagonals of a rhombus have lengths of 15 and 20 . Find its area. If an altitude of the rhombus is 12 , find the length of one side.

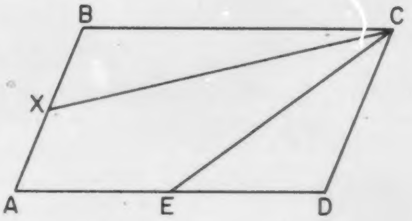
20. Find the area of the polygonal-region which is the intersection of $\{(x,y)\colon -7 \leq x \leq 5\}$ and $\{(x,y)\colon -5 \leq y \leq -1\}$. (Draw and shade the polygonal-region.)

21. Diagonal $\overline{AD}$ of the pentagon ABCDE shown has length 44 and the perpendiculars from B , C , and E have lengths 24 , 16 , and 15 , respectively. AB = 25 and CD = 20 . What is the area of the pentagon?

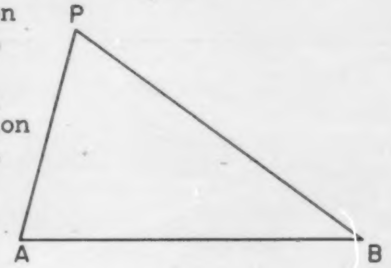


22. Given: Parallelogram ABCD with X and E midpoints of $\overline{AB}$ and $\overline{AD}$ respectively

To prove: Area of AECX is $\frac{1}{2}$ the area of ABCD .



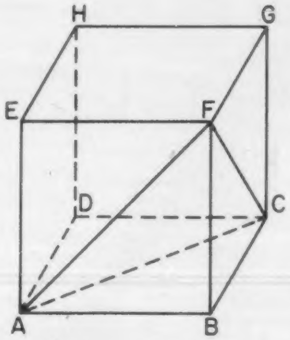
23. If $\overline{AB}$ is a segment in plane $\mathcal{E}$, what other positions of P in plane $\mathcal{E}$ will let the area of $\triangle ABP$ remain constant? Describe the location of all possible positions of P in plane $\mathcal{E}$ which satisfy the condition. Describe the location of all possible positions of P in space which satisfy the condition.



24. This figure represents a cube. The plane determined by points A , C and F is shown. If $\overline{AB}$ is 9 inches long, how long is $\overline{AC}$?

What is the measure of $\angle$ FAC ?

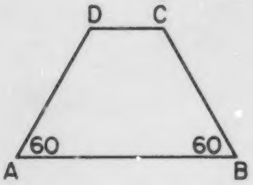
What is the area of $\triangle$FAC ?



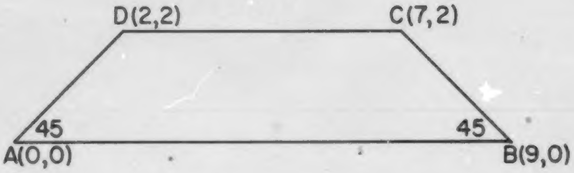
25. Find the length of the diagonal of a cube whose edge is 6 units long.

26. Explain how to cut a polygonal-region bounded by a trapezoid into two polygonal-regions having equal areas by a line through a vertex of the trapezoid.

27. In trapezoid ABCD , base angles of measure 60 include a base of length 12 . The non-parallel side $\overline{AD}$ has length 8 . Find the area of the trapezoid.



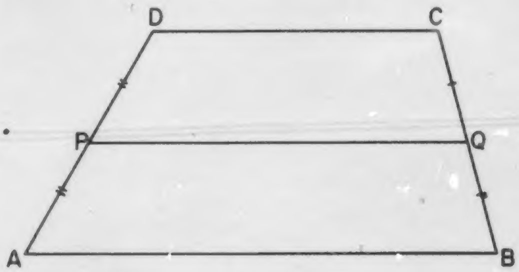
28. Find the area of trapezoid ABCD .



29. (a) Prove the following theorem: The median of a trapezoid is parallel to the bases and equal in length to half the sum of the lengths of the bases.

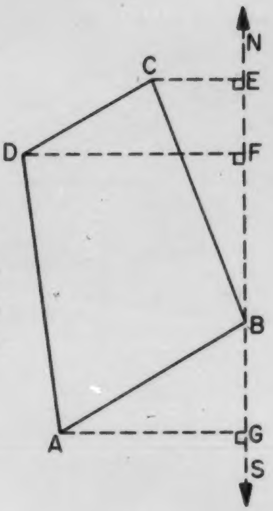
(b) If AB = 9 and DC = 7 , then PQ = ______ .

(c) If DC = $5\frac{1}{2}$ and PQ = 7 , then AB = ______ .



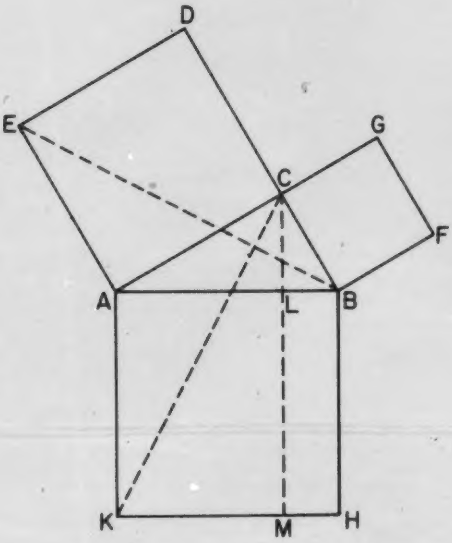
30. In surveying the field ABCD shown here, a surveyor laid off north-and-south line $\overleftrightarrow{NS}$ through B and then located the east-and-west lines $\overleftrightarrow{CE}$, $\overleftrightarrow{DF}$, $\overleftrightarrow{AG}$. He found that
CE = 5 (rods), DF = 12 (rods),
AG = 10 (rods), BG = 6 (rods),
BF = 9 (rods), FE = 4 (rods).

Find the area of the field.



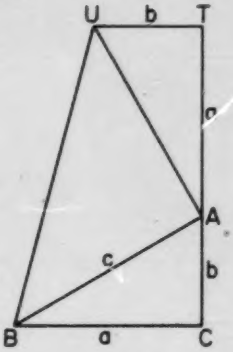
31. By hypothesis we have a right triangle ABC with the right angle at C. We are to prove that the area of square AKHB is equal to the sum of the areas of square ACDE and square BFGC.

Hint: Consider $\overline{CK}$, $\overline{BE}$, $\overline{CM}$ such that $\overline{CM} \parallel \overline{AK}$. Study $\triangle KAC$ and $\triangle BAE$. (Proof from Euclid.)



32. A proof of the Pythagorean Theorem by the following method is attributed to President Garfield. Let $\triangle ABC$ have right angle at C. On the ray opposite to $\overrightarrow{AC}$, let T be a point such that $AT = a$. Let U be a point on the same side of $\overleftrightarrow{AC}$ as B such that $\overline{TU} \perp \overline{AT}$ and $TU = b$.

Find AU, express the area of the trapezoid $BCTU$ in two ways, and deduce that $a^2 + b^2 = c^2$.



33. In a cube with A as one vertex, a triangular pyramid is formed by joining to A and to each other the midpoints of the three edges which meet at A. Find the total area of the pyramid if each side of the cube is 12.

34. A regular right hexagonal prism 10 units high has a lateral area of 480. Find the apothem of the base and the total area of the prism.

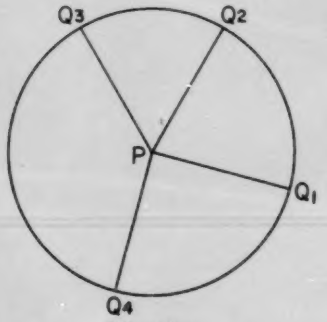
Chapter 12

CIRCLES AND SPHERES

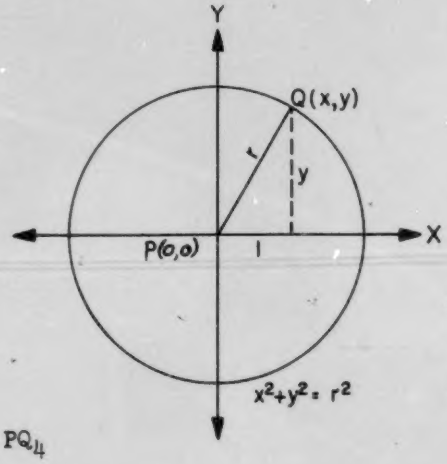
12-1. Basic Definitions.

In this chapter we begin the study of sets of points not made up of planes, halfplanes, lines, rays and segments. The simplest such curved figures are the circle and the sphere and portions of these. We begin with some definitions.

> DEFINITIONS. The set of all points in a plane whose distances from a given point in the plane are a given number is called a circle.
>
> The given point is called the center of the circle.
>
> The given number is called the radius of the circle.



Circle



$$PQ_1 = PQ_2 = PQ_3 = PQ_4$$

If we choose a two-dimensional coordinate system in the plane whose origin is the center P and if Q is any point of the circle, then $PQ = r$. Using the distance formula of Theorem 8-4, we can write

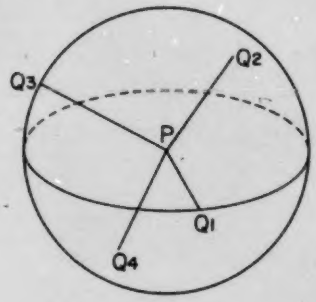
$$\sqrt{(x - 0)^2 + (y - 0)^2} = r \quad \text{or} \quad x^2 + y^2 = r^2 .$$

Therefore the circle is $\{(x,y) : x^2 + y^2 = r^2\}$.

DEFINITIONS. The set of all points in space whose distances from a given point are a given number is called a sphere.

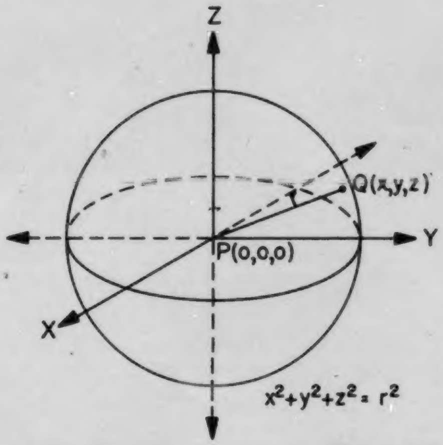
The given point is called the center of the sphere.

The given number is called the radius of the sphere.



Sphere

$PQ_1 = PQ_2 = PQ_3 = PQ_4$



If we choose a three-dimensional coordinate system whose origin is at the center, P , and Q is any point of the sphere, then $PQ = r$. Using the three-dimensional distance formula, we can write

$$\sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = r \quad \text{or} \quad x^2 + y^2 + z^2 = r^2 .$$

Therefore the sphere is $\{(x,y,z): x^2 + y^2 + z^2 = r^2\}$.

DEFINITION. Two or more spheres or circles with the same center are called concentric.

THEOREM 12-1. The intersection of a sphere with a plane through its center is a circle whose center and radius are the same as those of the sphere.

Proof: We choose a three-dimensional coordinate system with the center of the sphere P as origin $(0,0,0)$ and the given plane as the xy-plane (in which every point has zero as its z-coordinate). If r is the radius of the sphere, the intersection of the sphere and the plane is

$$\{(x,y,z): x^2 + y^2 + z^2 = r^2 \text{ and } z = 0\} .$$

We recognize this to be the set of points in the xy-plane given by

$$\{(x,y): x^2 + y^2 = r^2\} .$$

This is a circle whose center and radius are the same as those of the sphere.

DEFINITION. The circle of intersection of a sphere with a plane through the center is called a *great circle* of the sphere.

There are two types of segments that are associated with spheres and circles.

DEFINITIONS. A *chord* of a circle or a sphere is a segment whose endpoints are points of the circle or the sphere. The line containing a chord is a *secant*.

A *diameter* is a chord containing the center.

A *radius* is a segment one of whose endpoints is the center and the other one a point of the circle or the sphere.

The latter endpoint is called the *outer end* of the radius.

The plural of radius is *radii*.

Notice that the single word "radius" is used to mean two different objects--a certain segment and also the length of that segment. This should not be confusing because once we know that the word has two meanings we can easily decide which one is intended wherever the word occurs.

The word "diameter" also has two meanings. In addition to meaning a certain kind of chord it also is used to mean the length of such a chord.

DEFINITIONS. Circles with congruent radii are called congruent circles.

Spheres with congruent radii are called congruent spheres.

A direct outcome of these definitions are the following two theorems.

THEOREM 12-2. The radii of a circle or congruent circles, or of a sphere or congruent spheres, are congruent.

THEOREM 12-3. The diameters of a circle or congruent circles, or of a sphere or congruent spheres, are congruent.

It should be clear that the radii and diameters referred to in these theorems are segments, not numbers.

Problem Set 12-1

1. Fill in the blanks with a word or words which will best name or describe the indicated parts. Assume that points are where they appear to be.

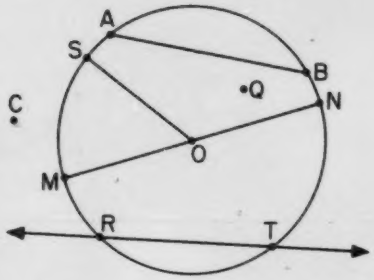


Diagram (a)

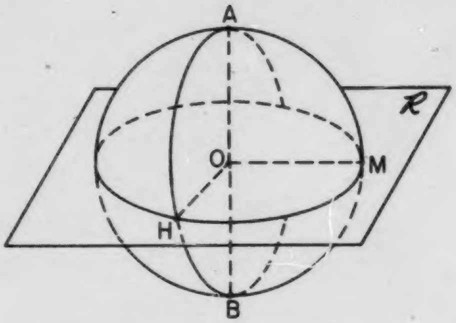


Diagram (b)

(a) Refer to Diagram (a).
Point O is the center of the circle.

(1) Any point X on any subset of the circle is a distance _____ from O .

(2) $\overline{OS}$ is a _____ .

(3) OS is the _____ .

(4) $\overline{MN}$ is a _____ , $\overleftrightarrow{MN}$ is a _____ .

(5) $\overline{RT}$ is a _____ , $\overleftrightarrow{RT}$ is a _____ .

(6) Points in the diagram which are in (on) the given circle are _____ .

(7) Points in the diagram which are not in the circle are _____ .

(8) S is the _____ of the radius _____ .

(9) Each point in the circle is the _____ of one and only one radius.

(10) Any point X on the subset of the circle between R and M would in every possible position be a distance _____ from O .

(b) Refer to Diagram (b).
Point O is the center of the sphere.

(1) $\overline{OA}$ is a _____ .

(2) Points _____ are outer endpoints of given radii.

(3) If O, A, and B are collinear, then $\overline{AB}$ is a _____ .

(4) If O lies in plane $\mathcal{R}$, then the circle with center O and radius $\overline{OM}$ in $\mathcal{R}$, is a _____ of the sphere.

(5) $\overline{BM}$ is a _____ .

(6) Every point on the _____ is the outer end of _____ radius.

(7) All points in $\mathcal{R}$ which lie at a distance OB from O lie in _____ with center _____ and radius _____ .

(8) How many planes may contain any given point such as O ? How many great circles are there on any given sphere? All great circles on a given sphere are _____ to each other.

(9) In order to specify a unique sphere, we must be given _______________ .

(10) With a given point as center, it is possible to consider (how many) spheres? All these spheres are called _____ spheres.

2. Tell whether the following statements are true or false.

(a) There is exactly one great circle of a sphere.

(b) Every chord of a circle contains two points of the circle.

(c) A radius of a circle is a chord of the circle.

(d) The center of a circle bisects only one of the chords of the circle.

(e) A secant of a circle may intersect the circle in only one point.

(f) All radii of a sphere are congruent.

(g) A chord of a sphere may be longer than a radius of the sphere.

(h) If a sphere and a circle have the same center and if they intersect, then the intersection is a circle.

3. Tell whether the following statements are true or false.

   (a) If a line intersects a circle in one point, it intersects the circle in two points.

   (b) The intersection of a line and a circle may be empty.

   (c) In the plane of a circle, a line which passes through the center of the circle has two points in common with the circle.

   (d) A circle and a line may have three points in common.

   (e) If a plane intersects a sphere in at least two points, the intersection is a line.

   (f) A plane cannot intersect a sphere in one point.

   (g) If two circles intersect, their intersection is two points.

   (h) The radius of a circle is a subset of the circle.

4. Consider an xy-coordinate system and yz-coordinate system as indicated in the diagrams below.

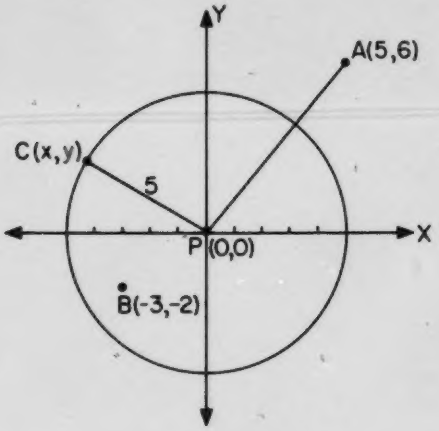


Figure (a)

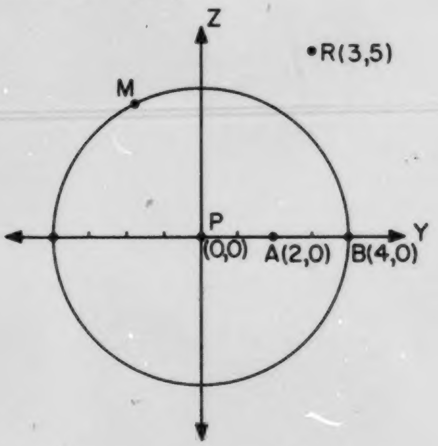


Figure (b)

(a) Refer to Figure (a).

(1) Express the distance between C and P in terms of x and y .

(2) Write an equation of the circle with center at P and radius 5 .

(3) Write the coordinates of 4 points which you know lie on the circle in Part (2).

(4) Find the distance between B and P .

(5) Write an equation of a circle which has P as a center and contains B .

(6) Write an equation of the circle which has P as its center and contains A .

(b) Use Figure (b).

(1) What is the radius of the circle which has P as center and contains B ?

(2) Write an equation of the given circle.

(3) Find the distance RP . How can you tell without a diagram that R is not on the circle?

(4) Write an equation of a circle with center P and which contains A .

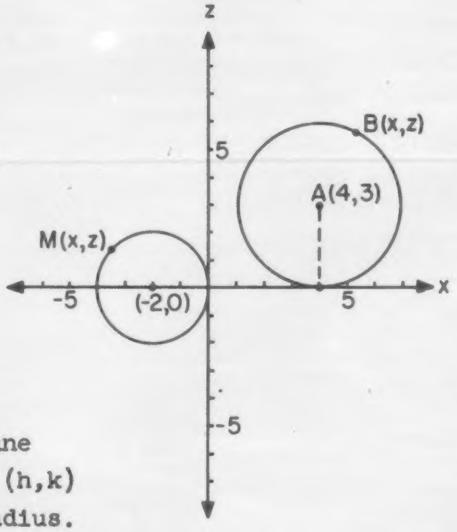
5. B is a point on the circle with center A and radius 3 . Use the xz-coordinate system as indicated.

(a) Express the distance between B and A .

(b) Write an equation of the circle which has A as center and radius 3 .

(c) Write an equation of the circle which has point (-2,0) as center and which contains the origin.

(d) Write an equation of a circle in the xz-plane with center at point (h,k) and with r as its radius.



6. Given circle $C = \{(x,y): x^2 + y^2 = 25\}$. Check whether or not the following points are points of C .

   (a) $(0,-5)$.

   (b) $(-3,4)$.

   (c) $(\sqrt{20},-\sqrt{5})$.

   (d) $(12,13)$.

7. Write an equation of the sphere with center at point $(0,0,0)$ and radius $= 3$.

8. Given sphere $S = \{(x,y,z): x^2 + y^2 + z^2 = 169\}$. Check whether or not the following points are points of S .

   (a) $(0,13,0)$.

   (b) $(-3,4,-12)$.

   (c) $(4,-12,3)$.

   (d) $(0,0,0)$.

   (e) $(\sqrt{168},1,0)$.

   (f) $(\sqrt{120},-\sqrt{40},3)$.

   (g) $(-\sqrt{108},2,-\sqrt{59})$.

9. Find 5 more points of S in Problem 8.

10. Using the set notation, write an expression for the points of the circle whose center is $(0,0)$ and whose radius is:

    (a) 3 . (b) $\frac{1}{2}$. (c) $\sqrt{5}$.

11. Given $C = \{(x,y): x^2 + y^2 = 25\}$.

    (a) What restriction on x and y would give only the portion of the circle in Quadrant I?

    (b) What portion of the circle would you be considering under the restriction, $x > 0$?

    (c) What restriction on x and y would give the intersection of C and Quadrant III?

12. Given $C = \{(x,y): x^2 + y^2 = 9\}$.

    (a) Find x if $(x,2)$ is a point of C .

    (b) Find y if $(3,y)$ is a point of C .

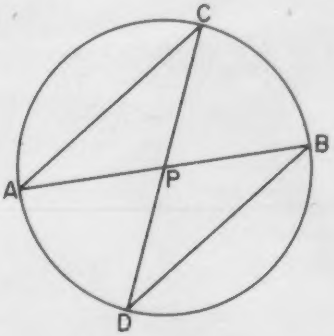
    (c) Can you find y so that $(4,y)$ is a point of C ? Explain.

13. Given $S = \{(x,y,z): x^2 + y^2 + z^2 = 25\}$.
    (a) Find z if $(3,0,z)$ is a point of S .
    (b) Find y if $(-4,y,3)$ is a point of S .
    (c) Find x if $(x,0,0)$ is a point of S .
    (d) Can you find z such that $(3,5,z)$ is a point of S ? Explain.

14. Prove: A diameter of a circle is its longest chord.

15. Given: $\overline{AB}$ and $\overline{CD}$ are diameters.

    Prove: $\overline{AC} \cong \overline{BD}$.



16. Prove: If $\overline{AB}$ and $\overline{CD}$ are distinct diameters of a circle, then ACBD is a rectangle.

17. Prove that the midray of the angle formed by two radii of a circle, $\overline{PA}$ and $\overline{PB}$, lies in the perpendicular bisector of $\overline{AB}$.

18. Consider a sphere whose center is at $O(2,3,-1)$. Let $Q(x,y,z)$ be a point of the sphere. What is the distance OQ , by the distance formula? Is $\overline{OQ}$ a radius of the sphere? Write an equation of the sphere which has (2,3-1) as center and which contains Q , if $OQ = 5$. (Eliminate radicals by squaring both members of the equation.)

12-2. Tangent Lines.

Anyone who looks at a drawing of a circle sees that it divides the plane into two regions, one consisting of the points inside the circle and the other of the outside points. We now define these terms formally.

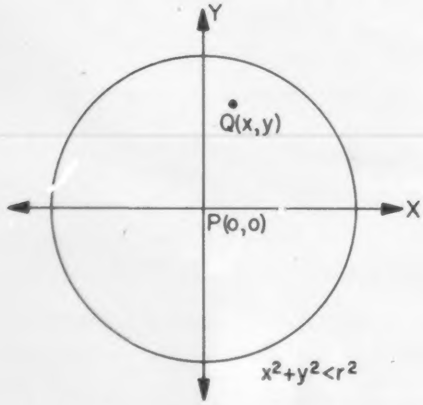
> DEFINITIONS. The interior of a circle is the set of all points in the plane of the circle whose distances from the center are less than the radius.
>
> The exterior of the circle is the set of all points in the plane of the circle whose distances from the center are greater than the radius.

From these definitions it follows that a point in the plane of a circle is either in the interior of the circle, on the circle, or in the exterior of the circle. (We frequently use the more common word "inside" for "in the interior of," etc.) In terms of an xy-coordinate system whose origin is the center of a given circle of radius r, the interior of the circle is

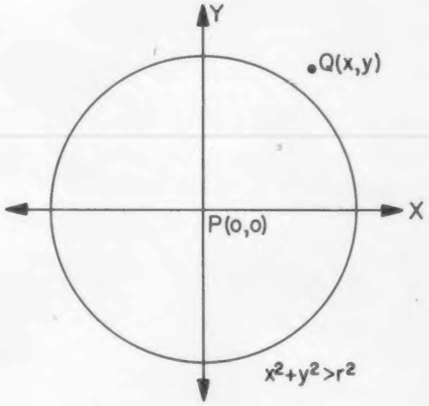
$$\{(x,y): x^2 + y^2 < r^2\}$$

and its exterior is

$$\{(x,y): x^2 + y^2 > r^2\} .$$



Q is an interior point of C .



Q is an exterior point of C .

Problem Set 12-2a

(Exploratory)

1. Given $C = \{(x,y): x^2 + y^2 = 16\}$ and $M = \{(x,y): x = a\}$. Find the set of points in the intersection of C and M, if $a = 3$; if $a = 4$; if $a = 5$.

2. Using the results you found in Problem 1, complete the following.

   (a) The intersection of C and M contains __?__ point(s) if $a < 4$.

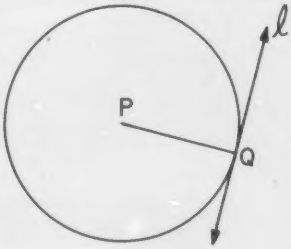
   (b) The intersection of C and M contains __?__ point(s) if $a = 4$.

   (c) The intersection of C and M contains __?__ point(s) if $a > 4$.

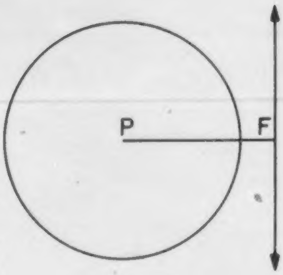
3. What three relations between a circle and a line in the plane of the circle are suggested by Problems 1 and 2?

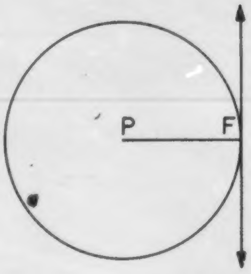
If a stone is twirled on the end of a string in a circular path and then let go, it will "fly off on a tangent." Try to see how this use of the word tangent is related to the one we now give.

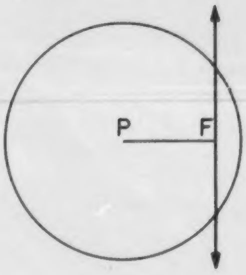
> *DEFINITIONS*. A *tangent* to a circle is a line in the plane of the circle which intersects the circle in only one point.
>
> This point is called the *point of tangency*, or *point of contact*, and we say that the line and the circle *are tangent* at this point.
>
> In the figure, ℓ is tangent to the circle at Q.



Lines and circles are important subsets of planes. Let us consider a single plane and study the relations of lines and circles to one another. It looks as if the following three figures indicate a complete catalog of the possibilities:







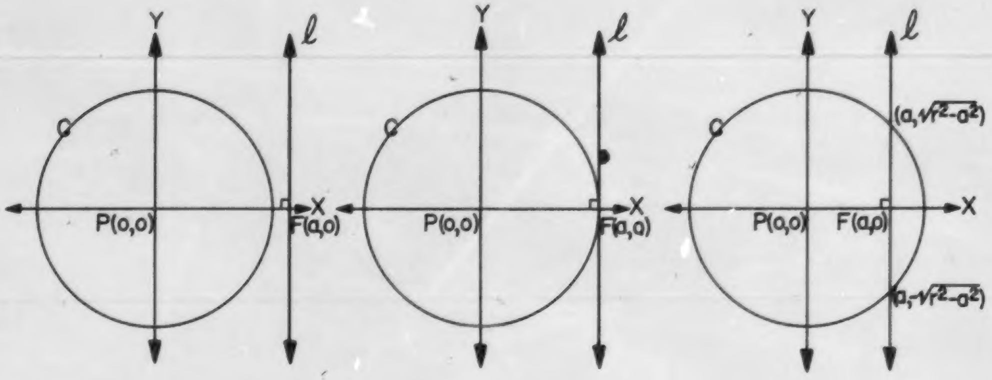
In each case, P is the center of the circle, and F is the foot of the perpendicular from P to the line. We shall soon see that this point F, the foot of the perpendicular, is the key to the whole situation. If F is outside the circle, as in the first figure, then all other points of the line are also outside, and the line and the circle do not intersect at all. If F is on the circle, then the line is a tangent line, as in the second figure, and the point of tangency is F. If F is inside the circle, as in the third figure, then the line is a secant, and the points of intersection are equidistant from the point F. To verify these statements, we need to prove the following theorem:

THEOREM 12-4. Given a line ℓ and a circle C in the same plane. Let P be the center of the circle, and let F be the foot of the perpendicular from P to the line.

(1) Every point of ℓ is outside C if and only if F is outside C.
(2) ℓ is a tangent to C if and only if F is on C.
(3) ℓ is a secant of C if and only if F is inside C

Proof: Let r be the radius of C and let $PF = a$.

We introduce the xy-coordinate system with origin at P whose y-axis is parallel to ℓ and whose positive x-axis contains F. Then $F = (a,0)$, $C = \{(x,y): x^2 + y^2 = r^2\}$ and $\ell = \{(x,y): x = a\}$ or $\{(a,y)\}$.



(1) F is outside C. (2) F is on C. (3) F is inside C.

(1) Suppose F is outside C, then $a > r$. Since a and r are positive numbers, it follows that $a^2 > r^2$ and $a^2 + y^2 > r^2$. Therefore all points (a,y) are outside C. Since $\ell = \{(a,y)\}$, then all points of ℓ are outside C.

Of course, if every point of ℓ is outside C, then F is also outside C. This proves both parts of (1).

(2) Suppose F is on C. Then $r = a$, and the intersection of ℓ and C is $\{(x,y): x^2 + y^2 = a^2 \text{ and } x = a\}$, or $\{(a,y): y^2 = 0\}$.

But there is exactly one number whose square is zero, namely zero. Therefore the only point of intersection of ℓ and C is $F(a,0)$. Therefore ℓ is a tangent to C.

If ℓ is a tangent, it can have only one point in common with C. That point is shown to be $F(a,0)$. Thus, both parts of (2) are proved.

(3) Suppose F is inside C , then $r > a$. The intersection of ℓ and C is $\{(x,y): x^2 + y^2 = r^2 , x = a$ and $a < r\}$ or $\{(a,y): y^2 = r^2 - a^2 , r > a\}$. Since $r^2 - a^2$ is a positive number, y can be either $\sqrt{r^2 - a^2}$ or $-\sqrt{r^2 - a^2}$. Therefore the intersection consists of $(a, \sqrt{r^2 - a^2})$ and $(a, -\sqrt{r^2 - a^2})$. These are distinct points. Why? Therefore ℓ is a secant.

If ℓ is a secant, it intersects C in two distinct points, which we have shown to be $(a, \sqrt{r^2 - a^2})$ and $(a, -\sqrt{r^2 - a^2})$. This implies that $r^2 - a^2 > 0$. (Why can't $r^2 - a^2 = 0$?) . And because r and a are positive, $r > a$. But $PF = a$; therefore $PF < r$, or F is inside C . This completes the proof of the theorem.

The following table displays some of the facts about F that we met in our proof.

	Case 1	Case 2	Case 3
F(a,0)	$a > r$ $a^2 + y^2 > r^2$ no y for which $a^2 + y^2 = r^2$. No point of ℓ lies on C .	$a = r$ $a^2 + y^2 = r^2$ if and only if $y = 0$. Only F is on C .	$a < r$ $a^2 + y^2 < r^2$ if and only if $y = r^2 - a^2$ or $y = -\sqrt{r^2 - a^2}$. ℓ and C have exactly two points in common. They are $(a, \sqrt{r^2 - a^2})$ and $(a, -\sqrt{r^2 - a^2})$.

Now we can proceed to our first basic theorems on tangents and chords which are all corollaries of Theorem 12-4. To prove them, you merely need to refer to Theorem 12-4 and see which of the three cases of the theorem applies.

Corollary 12-4-1. Given a circle and a coplanar line, the line is a tangent to the circle if and only if it is perpendicular to a radius of the circle at its outer end.

Corollary 12-4-2. A diameter of a circle bisects a non-diameter chord of the circle if and only if it is perpendicular to the chord.

Corollary 12-4-3. In the plane of a circle, the perpendicular bisector of a chord contains the center of the circle.

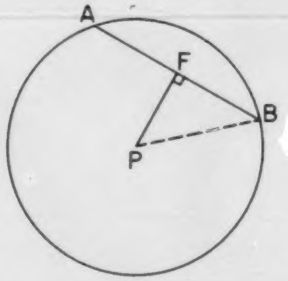
Hint for proof: Use Corollary 12-4-2.

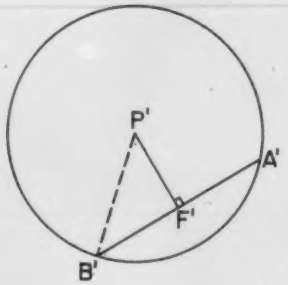
Corollary 12-4-4. If a line in the plane of a circle intersects the interior of the circle, then it intersects the circle in exactly two points.

Case (3) applies. [In Case (1) and (2), the line does not intersect the interior of the circle.]

THEOREM 12-5. Chords of congruent circles are congruent if and only if they are equidistant from the centers.

Proof: Let P and P' be the centers of the congruent circles, let $\overline{AB}$ and $\overline{A'B'}$ be the chords, let F be the foot of the perpendicular from P on $\overleftrightarrow{AB}$ and let F' be the foot of the perpendicular from P' on $\overleftrightarrow{A'B'}$. Then by Corollary 12-4-2, we have $FB = \frac{1}{2}AB$ and $F'B' = \frac{1}{2}A'B'$.





By the Pythagorean Theorem

$$(PB)^2 = (PF)^2 + (FB)^2 \qquad (P'B')^2 = (P'F')^2 + (F'B')^2$$

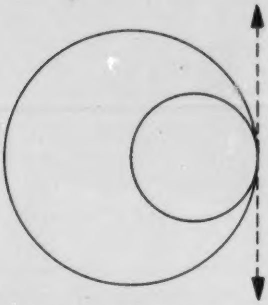
$$= (PF)^2 + (\tfrac{1}{2}AB)^2 \qquad = (P'F')^2 + (\tfrac{1}{2}A'B')^2 .$$

By hypothesis $PB = P'B'$ so

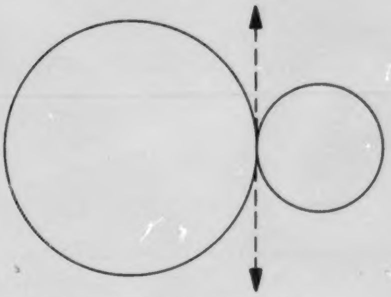
$$(PF)^2 + (\tfrac{1}{2}AB)^2 = (P'F')^2 + (\tfrac{1}{2}A'B')^2 .$$

It follows that $PF = P'F'$ if and only if $AB = A'B'$.

DEFINITIONS. Two circles are tangent if and only if they are coplanar and tangent to the same line at the same point. Tangent circles are internally or externally tangent accordingly as their centers lie on the same side or on opposite sides of the common tangent line.



Internally tangent



Externally tangent

Problem Set 12-2b

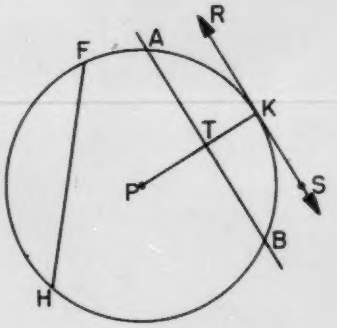
1. Given: $C = \{(x,y): x^2 + y^2 = 36\}$. Tell whether each of the following points is in the interior, on, or in the exterior of C .

   (a) $(-6,0)$.
   (b) $(-6,1)$.
   (c) $(-6,-1)$.
   (d) $(5,2)$.
   (e) $(-\sqrt{27},-3)$.
   (f) $(5,5)$.
   (g) $(4,-4)$.

2. Given: $(3,5)$ is on the circle whose center is $(0,0)$.

   (a) Find the radius of this circle.
   (b) Find four points on the circle.
   (c) Find two points in the interior of the circle.
   (d) Find two points in the exterior of the circle.

3. State the number of the theorem or corollary which justifies each conclusion below. P is the center of a circle. F, H, B, K, A are points on the circle and S, T, and R are coplanar with the circle.



(a) If $TA = TB$, then $\overline{PK} \perp \overline{AB}$.

(b) If $\overleftrightarrow{RK} \perp \overline{PK}$, then $\overleftrightarrow{RK}$ is tangent to the circle.

(c) If T is in the interior of the circle, then $\overleftrightarrow{KP}$ will intersect the circle in exactly one point other than point K.

(d) The perpendicular bisector of $\overline{FH}$ contains P.

(e) If $\overline{AB}$ and $\overline{FH}$ are equidistant from P, then $\overline{AB} \cong \overline{FH}$.

(f) If $\overleftrightarrow{RS}$ is tangent to the circle, then $\overline{PK} \perp \overleftrightarrow{RS}$.

(g) If $\overline{PK} \perp \overline{AB}$, then $AT = TB$.

(h) If $\overline{AB} \cong \overline{FH}$, then $\overline{AB}$ and $\overline{FH}$ are equidistant from P.

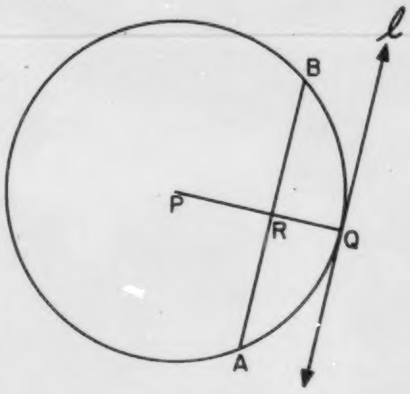
4. In a circle with radius of 5 units, how long is a chord 3 units from the center of the circle?

5. If a chord 4 inches long is 1.5 inches from the center of a circle, what is the radius of the circle?

6. How far from the center of a circle with radius equal to 12 is a chord whose length is 8?

7. Chord $\overline{AB}$ is parallel to ℓ which is tangent to the circle at Q. P is the center of the circle. $\overline{AB}$ bisects $\overline{PQ}$ at R. $AB = 12$.

Find PQ.



8. Given: The figure below, with C the center of the circle and $\overline{KT} \perp \overline{RS}$. In the ten problems respond as follows:

   Write "A" if more numerical information is given than is needed to solve the problem.

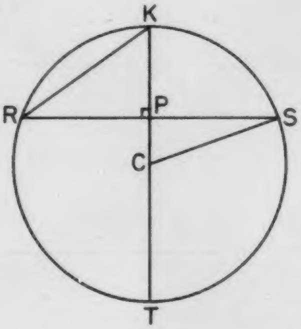
   Write "B" if there is insufficient information to solve the problem.

   Write "C" if the information is sufficient and there is no unnecessary information.

   Write "D" if the information given is contradictory.

   (You do not need to do the computations.)

   (a) KP = 4 , PC = 1 , CT = 6 , KT = ?
   (b) RP = 5 , RS = ?
   (c) CT = 13 , CP = 5 , RS = ?
   (d) KP = 18 , RS = 48 , KC = 25 , RK = ?
   (e) PC = 3.5 , RS = 24 , RK = ?
   (f) KT = 40 , RP = 16 , CS = ?
   (g) CS = 8 , TK = 16 , PC = ?
   (h) RK = 20 , RS = 32 , KP = 13 , KT = ?
   (i) RS = 6 , KC = 5 , PT = ?
   (j) PT = 5 , CS = 6 , RS = ?

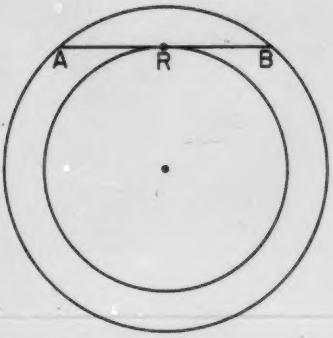


9. In a circle whose diameter is 30 inches a chord is drawn perpendicular to a radius at a point on the radius 3 inches from its outer end. Find the length of the chord.

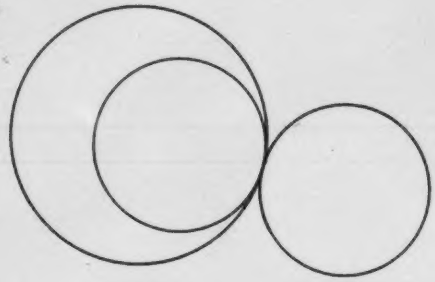
10. Prove that the tangents to a circle at the ends of a diameter are parallel.

11. Write Corollary 12-4-2 as two statements, each the converse of the other. Prove each.

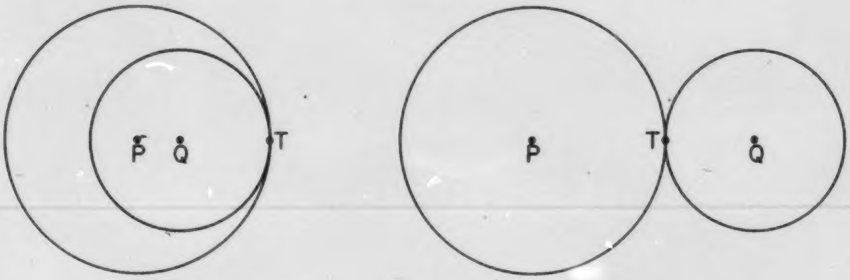
12. For the concentric circles of the figure, prove that all chords of the larger circle which are tangent to the smaller circle are bisected at the point of contact.



13. One arrangement of three circles so that any one is tangent to each of the other two is shown here. Make sketches to show three other arrangements of three circles with each circle tangent to each of the other two.

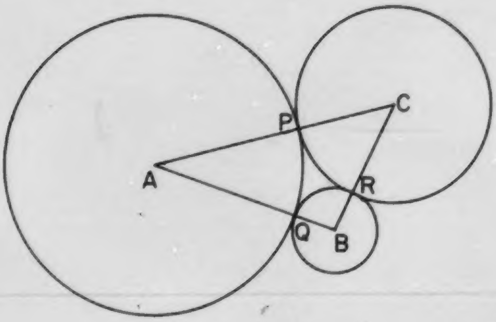


14. Prove: The line of centers of two tangent circles contains the point of tangency. (Hint: Draw the common tangent.)



Case I Case II

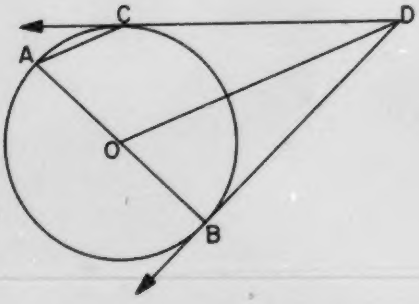
15. In the figure, A, B and C are the centers of the circles. $AB = 14$, $BC = 10$, $AC = 18$. Find the radius of each circle.



16. Prove: The midpoints of all chords congruent to a given chord in a given circle lie on a circle concentric with the original circle and with a radius equal to the distance of a chord from the center; and the chords are all tangent to this inner circle.

17. (a) The distance from P, the center of a circle, to T, an exterior point, is 20. A tangent from T to the circle has a point of contact A. If the radius of the circle is 12, find AT.
    (b) A second tangent from T has B as a point of contact. Find AB.

18. In the circle with center at O, $\overline{AB}$ is a diameter and $\overline{AC}$ is any other chord from A. If $\overleftrightarrow{CD}$ is the tangent at C, and $\overleftrightarrow{DO} \parallel \overleftrightarrow{AC}$, prove that $\overleftrightarrow{DB}$ is tangent at B.



19. Consider the circle $C = \{(x,y): x^2 + y^2 = 100\}$.
    (a) If line ℓ, $\{(x,y): x = a\}$, is tangent to circle C, find the values for a.
    (b) Find an equation for a line t tangent to circle C at $T(5\sqrt{2}, 5\sqrt{2})$.

(Hints: What is the slope of the radius to t ?
What must be the slope of t ?
Must t contain $(5\sqrt{2}, 5\sqrt{2})$?)

20. Consider the set $P = \{(x,y): (x - 1)^2 + (y + 2)^2 = 25\}$.

    (a) Can you interpret the equation as specifying that the distance between (x,y) and $(1,-2)$ is 5 ?

    (b) Is the set a circle? If so, what are the coordinates of its center and the length of its radius?

    (c) Given $P = \{(x,y): (x^2 - 2x + 1) + (y^2 + 4y + 4) = 25\}$
    Show that $P = \{(x,y): x^2 + y^2 - 2x + 4y = 20\}$.
    If you were confronted with this last equation, could you complete the squares to reproduce the original? Demonstrate this process.

    (d) Find an equation of a line t which is tangent to the circle P at the point $(5,1)$. (Hint: Find the slope of the radius to $(5,1)$. Use its negative reciprocal as the slope of t . Tangent t must contain $(5,1)$.)

21. Consider the circle $C = \{(x,y): x^2 + y^2 = 1\}$.

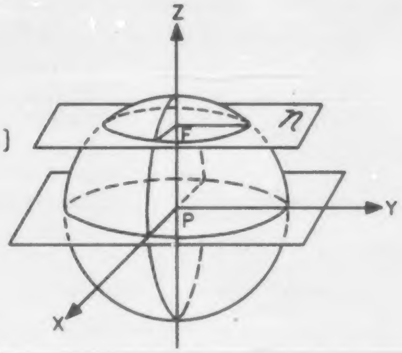
    (a) Write an equation of the line tangent to C which contains the point $(-1,0)$.

    (b) Write an equation of the tangent line to C which contains the point $T(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. (Hint: Is the tangent line perpendicular to the radius of C which contains T ?)

    (c) Find the coordinates of the point P on the x-axis which contains the tangent to T determined in (b) above.

    (d) Find the distance PT .

*22. Consider the sphere S and the plane $\mathcal{N}$ such that
$S = \{(x,y,z): x^2 + y^2 + z^2 = 25\}$
$\mathcal{N} = \{(x,y,z): z = a\}$.
How is $\mathcal{N}$ related to the xy-plane?
How is $\mathcal{N}$ related to the z-axis?



$\mathcal{N}$ intersects the z-axis at a point, say F , with coordinates $(0,0,a)$. Consider the intersection of S and $\mathcal{N}$, when a has the values indicated below.

(a) Assume $a = 4$.
What geometric figure is this intersection?

(b) Assume $a = 5$.
How many points are in this intersection?

(c) Assume $a = 7$.
How many points are in this intersection?

(d) What appears to be the relation between the intersection of S and $\mathcal{N}$ and the distance PF ?

12-3. Tangent Planes.

We have just studied circles and lines in a plane. We are now going to study spheres and planes in space. We shall see that many of the definitions and theorems of the last section resemble the definitions and theorems about spheres and planes.

DEFINITIONS. The interior of a sphere is the set of of all points whose distances from the center are less than the radius.

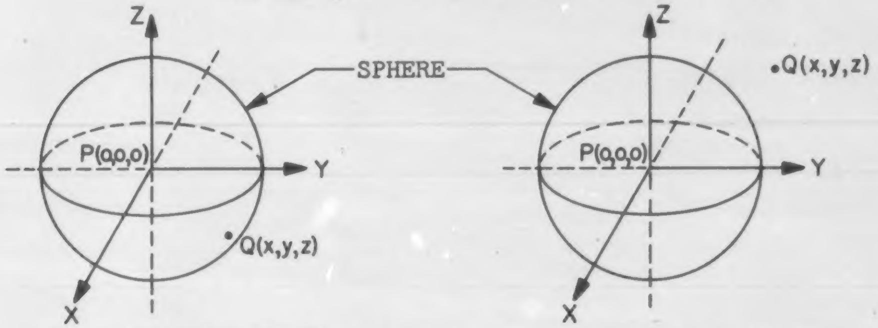
The exterior of the sphere is the set of all points whose distances from the center are greater than the radius.

In terms of a coordinate system whose origin is the center of a given sphere of radius r, the interior of the sphere is

$$\{(x,y,z)\colon x^2 + y^2 + z^2 < r^2\}$$

and its exterior is

$$\{(x,y,z)\colon x^2 + y^2 + z^2 > r^2\} .$$



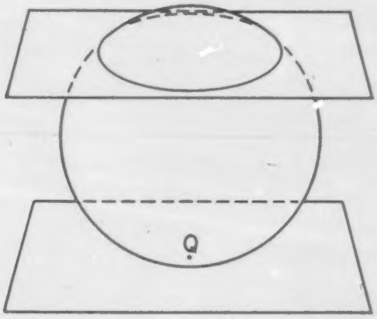
Q is an interior point $\quad x^2 + y^2 + z^2 < r^2$

Q is an exterior point $\quad x^2 + y^2 + z^2 > r^2$

DEFINITIONS. A plane that intersects a sphere in exactly one point is called a tangent plane to the sphere.

If the tangent plane intersects the sphere in the point Q then we say that the plane is tangent to the sphere at Q.

Q is called the point of tangency, or the point of contact.



When we investigated circle-line relations in Theorem 12-4 we found that the key in each phase of the study was the foot of the perpendicular from the center to the line. Our sphere-plane study also has a key. It is the foot of the perpendicular from the center of the sphere to the plane.

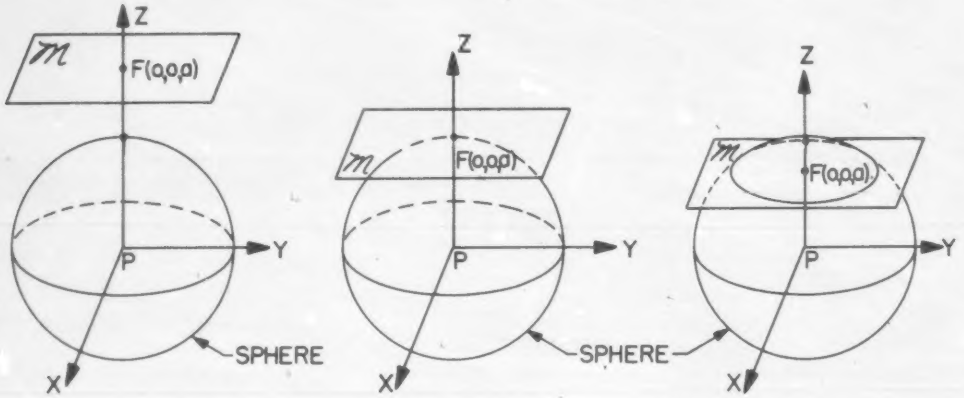
The basic theorem relating spheres and planes is the following:

THEOREM 12-6. Given a plane $\mathcal{M}$ and a sphere S with center P . Let F be the foot of the perpendicular from P to $\mathcal{M}$.

1. Every point of $\mathcal{M}$ is outside S if and only if F is outside S .
2. $\mathcal{M}$ is tangent to S if and only if F is on S .
3. $\mathcal{M}$ intersects S in a circle with center F if and only if F is inside S .

Proof: Let r be the radius of S and let PF = a .

We introduce an xyz-coordinate system with origin at P , whose xy-plane is parallel to $\mathcal{M}$ and whose positive z-axis contains F. Then $F = (0,0,a)$, $C = \{(x,y,z): x^2 + y^2 + z^2 = r^2\}$ and $\mathcal{M} = \{(x,y,z): z = a\}$ or $\{x,y,a)\}$.



(1) F is outside S . (2) F is on S . (3) F is inside S .

(1) Suppose F is outside S, then $a > r$. It follows, since a and r are positive numbers, that $a^2 > r^2$ and $x^2 + y^2 + a^2 > r^2$. This tells us that (x,y,a) is outside S. But $\mathcal{M} = \{(x,y,a)\}$. Therefore $\mathcal{M}$ is outside S.

Of course, if every point of $\mathcal{M}$ is outside S, then F is also outside S. This proves both parts of (1).

(2) Suppose F is on S. Then $r = a$ and the intersection of $\mathcal{M}$ and S is

$$\{(x,y,z): x^2 + y^2 + z^2 = a^2 \text{ and } z = a\} \text{ or}$$

$$\{(x,y,a): x^2 + y^2 = 0\}.$$

But there is only one pair of numbers, (x,y), namely $(0,0)$, such that $x^2 + y^2 = 0$. Therefore $\mathcal{M}$ and S have only $F(0,0,a)$ in common and it follows that $\mathcal{M}$ is tangent to S.

If $\mathcal{M}$ is tangent to S, they have only one point in common and it has been shown that $(0,0,a)$ is that point. Thus both parts of (2) are proved.

(3) Suppose F is inside C, then $r > a$. The intersection of $\mathcal{M}$ and S is

$$\{(x,y,z): x^2 + y^2 + z^2 = r^2,\ z = a,\ r > a\} \text{ or}$$

$$\{(x,y,a): x^2 + y^2 = r^2 - a^2,\ r > a\}.$$

Because $r^2 - a^2 > 0$, we can see from the form of the equation $x^2 + y^2 = r^2 - a^2$, that we have a circle in the plane $z = a$, with center $(0,0,a)$ and radius $\sqrt{r^2 - a^2}$.

On the other hand, if $\mathcal{M}$ intersects S it follows that $x^2 + y^2 = r^2 - a^2$ has a solution. This implies that $r^2 > a^2$ or $r > a$. Therefore $F(0,0,a)$ is in S.

This completes the proof of the theorem.

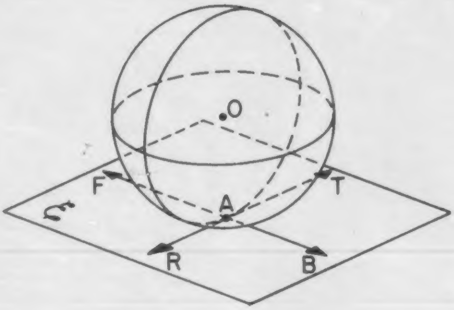
Corollary 12-6-1. A plane is tangent to a sphere if and only if it is perpendicular to a radius at its outer endpoint.

Corollary 12-6-2. A perpendicular from the center of a sphere to a chord of the sphere bisects the chord.

Corollary 12-6-3. The segment joining the center of a sphere to the midpoint of a chord is perpendicular to the chord.

Problem Set 12-3

1. The sphere with center O is tangent to plane $\mathcal{E}$ at A . $\overleftrightarrow{FB}$ and $\overleftrightarrow{RT}$ are lines of $\mathcal{E}$ through A . What is the relationship of $\overleftrightarrow{OA}$ to $\overleftrightarrow{FB}$ and $\overleftrightarrow{RT}$?



2. In a sphere having radius 10 , a segment from the center to the midpoint of a chord has length 6 . How long is the chord?

3. A sphere has radius 5 . A plane 3 units from the center intersects the sphere in a circle. What is the radius of this circle?

4. Prove that circles on a sphere in planes equidistant from the center of the sphere are congruent.

5. State Corollary 12-6-1 as two statements which are converses of each other. Prove each.

6. Show that two great circles of a sphere intersect at the endpoints of a diameter of the sphere.

7. Consider the sphere $S = \{(x,y,z): x^2 + y^2 + z^2 = 9\}$.

   (a) What is the center of S ? What is the radius of S ?

   (b) Write an equation of a plane tangent to S and parallel to the xz-plane. How many such planes are there?

   (c) Write equations of all planes tangent to S and parallel to the yz-plane.

8. Consider the sets:

$S = \{(x,y,z): x^2 + y^2 + z^2 = 16\}$

$M = \{(x,y,z): |x| > 4\}$; $N = \{(x,y,z): |y| > 4\}$

$R = \{(x,y,z): |z| > 4\}$

$T = \{(x,y,z): |x| < 4 , |y| < 4 , |z| < 4\}$.

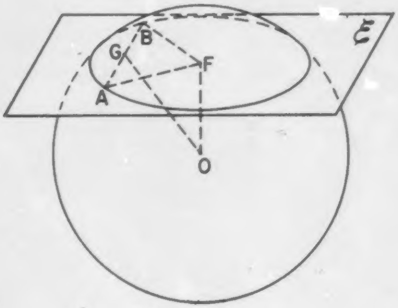
(a) Describe the intersection of S and M ; of S and N ; of S and R .

(b) Describe the set T.

(c) What is the intersection of S and T ?

9. Two great circles are said to be perpendicular if they lie in perpendicular planes. Show that, given any two great circles, there is one other great circle perpendicular to both. If two great circles on the earth are meridians (through the north and south poles) what great circle is their common perpendicular?

10. Plane $\mathcal{E}$ intersects a sphere whose center is O . A and B are two points of the intersection. F lies in plane $\mathcal{E}$. $\overleftrightarrow{OF} \perp \mathcal{E}$. $\overline{AF} \perp \overline{BF}$. If $AB = 5$ and $OF = AF$, find the radius of the sphere and $m\angle AOB$. If G is the midpoint of $\overline{AB}$, find OG .



11. Given a sphere and three points on it. Describe the steps you would take to find the center and the radius of the sphere.

12. Plane $\mathcal{E}$ is tangent to a sphere X at point T , and plane $\mathcal{F}$ is any plane other than $\mathcal{E}$ which contains T . Prove:

(a) that plane $\mathcal{F}$ intersects sphere S and plane $\mathcal{E}$ in a circle and a line respectively;

(b) that the line of intersection is tangent to the circle of intersection.

13. Consider the sphere $S = \{(x,y,z): x^2 + y^2 + z^2 = 100\}$.

(a) Find the intersection of sphere S with the plane $\{(x,y,z): z = 10\}$.

(b) Consider the plane $P = \{(x,y,z): z = 8\}$. In order for a point (x,y,z) to be in the intersection of S and P , certainly $z = 8$; what conditions, then, must x and y meet?

14. Consider the circle $C = \{(x,y): x^2 + y^2 - 4x + 6y = 23\}$.

(a) Complete squares and transform the equation of C to the form $(x - h)^2 + (y - k)^2 = r^2$. What are the coordinates of the center and the length of the radius of C ?

(b) Write an equation of the sphere S whose radius has the same length as the radius of C and whose center is at $(2,-3,0)$.

(c) Write an equation describing a plane tangent to sphere S and perpendicular to the z-axis. (Two answers are possible.)

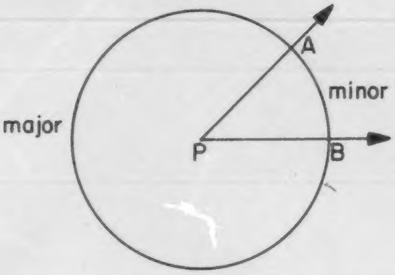
12-4. Arcs of Circles.

So far in this chapter we have been able to treat circles and spheres in a similar manner. For the rest of this chapter we confine ourselves exclusively to circles. The topics we discuss have their corresponding analogies in the theory of spheres but these are too complicated to consider in a beginning course.

DEFINITION. A central angle of a given circle is an angle whose vertex is the center of the circle.

DEFINITIONS. If A and B are two points of a circle with center P and if A and B are not the endpoints of a diameter of that circle, then the union of A, B, and all the points of the circle in the interior of ∠APB is a minor arc of the circle.

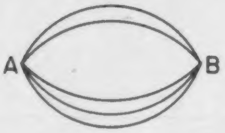
The union of A, B, and all points of the circle in the exterior of ∠APB is a major arc of the circle.



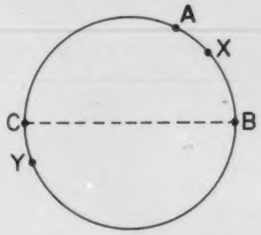
If $\overline{AB}$ is a diameter, the union of A, B, and all points of the circle in one of the two halfplanes, with edge $\overleftrightarrow{AB}$, lying in the plane of the circle is a semicircle.

An arc is either a minor arc, a major arc, or a semi-circle. A and B are the endpoints of the arc.

In some ways an arc of a circle is like a segment of a line; for instance, it has two endpoints. However, unlike the segment, an arc is not determined by its endpoints. In fact there are infinitely many arcs which have any given pair of points as their endpoints, as the figure suggests. This makes



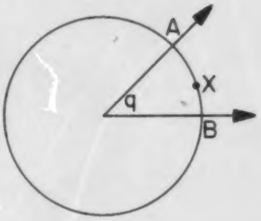
it hard to find a symbol to denote an arc, built up from the symbols for its endpoints. In spite of this we often denote an arc whose endpoints are A and B by $\widehat{AB}$. We must be sure we know what circle we have in mind for this to make sense, and also we must know which of the two arcs on that circle we have in mind. Sometimes it will be plain from the context which arc is meant. If not, we will pick another point X somewhere in the arc $\widehat{AB}$, and denote the arc by $\widehat{AXB}$. For example, in the figure, $\widehat{AXB}$ is a minor arc; $\widehat{AYB}$ is the associated major arc; and the arcs $\widehat{CAB}$ and $\widehat{CYB}$ are semicircles.



The reason for the names "minor" and "major" is apparent when one draws several arcs of each kind. In such drawings the major arc looks "bigger" than a minor arc. This relation will be made more precise in our next definition.

DEFINITION. If $\widehat{AXB}$ is any arc then its degree measure, $m\widehat{AXB}$, is given as follows:

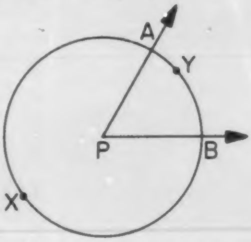
1. If $\widehat{AXB}$ is a minor arc, then $m\widehat{AXB}$ is the measure of the associated central angle.



$m\widehat{AXB} = m\angle q$

2. If $\widehat{AXB}$ is a semicircle, then $m\widehat{AXB} = 180$.
3. If $\widehat{AXB}$ is a major arc, and $\widehat{AYB}$ is the corresponding minor arc, then

   $m\widehat{AXB} = 360 - m\widehat{AYB}$.

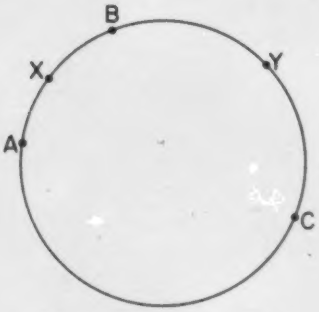


In the figure, m $\angle APB$ is 60 . Therefore $m\widehat{AYB}$ is 60 , and $m\widehat{AXB}$ is 300 .

Hereafter, $m\widehat{AXB}$ will be called simply the measure of the arc $\widehat{AXB}$. Note that an arc is minor or major according as its measure is less than or greater than 180 .

If X is a point of an arc $\widehat{AB}$ different from A and B , it determines, with A and B , two other arcs, $\widehat{AX}$ and $\widehat{XB}$. It is natural to inquire how the measures of such arcs, $\widehat{AX}$ and $\widehat{XB}$, are related to the measure of $\widehat{AB}$; the answer is simple and reasonable, namely, $m\widehat{AXB} = m\widehat{AX} + m\widehat{XB}$. This can actually be proved as a theorem but the proof is surprisingly long and tedious. We prefer to state the result as a postulate.

Postulate 30. If $\widehat{AB}$ and $\widehat{BC}$ are arcs of the same circle having only the point B in common, and if their union is an arc $\widehat{AC}$, then $m\widehat{AB} + m\widehat{BC} = m\widehat{AC}$.



$m\widehat{AXB} + m\widehat{BYC} = m\widehat{ABC}$.

Notice that for the cases in which $\widehat{AC}$ is a minor arc or a semicircle the theorem follows from the Protractor Postulate. It is the other case whose proof is difficult.

In each of the figures below, the angle x is said to be inscribed in the arc $\widehat{ABC}$.

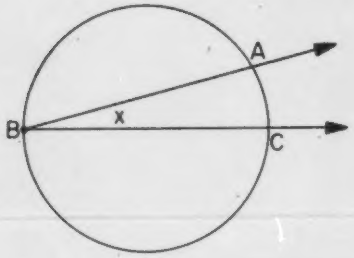


Figure a

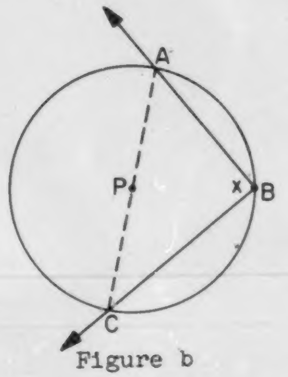


Figure b

DEFINITION. An angle is inscribed in an arc if and only if (1) the angle contains the two endpoints of the arc and (2) the vertex of the angle is a point, but not an endpoint, of the arc.

More concisely, $\angle ABC$ is inscribed in $\widehat{ABC}$.

In Figure a, the angle is inscribed in a major arc, and in Figure b, the angle is inscribed in a semicircle.

In each of the figures below, the angle shown is said to intercept $\widehat{PQR}$.

FIGURE c

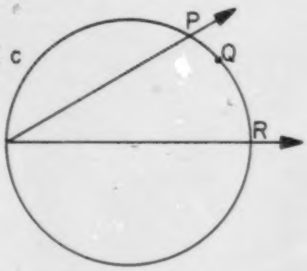


FIGURE d

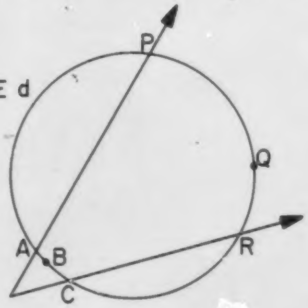


FIGURE e

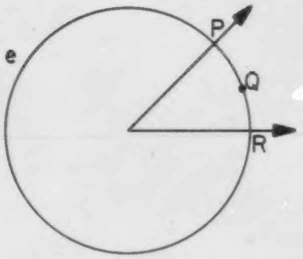
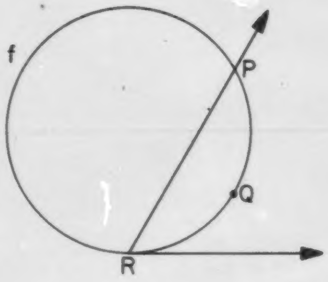


FIGURE f

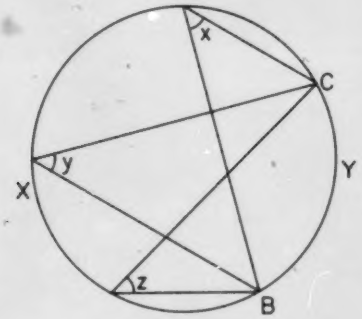


In Figure c, the angle is inscribed; in Figure d, the vertex is outside the circle; in Figure e, the angle is a central angle; and in Figure f, one side of the angle is tangent to the circle. In Figure d, the angle shown intercepts not only $\widehat{PQR}$ but also $\widehat{ABC}$.

These figures give the general idea of an intercepted arc. We will now define what it means to say that an angle intercepts an arc. You should check carefully to make sure that the definition really takes care of all four of the above cases.

> DEFINITION. An angle intercepts an arc if (1) the endpoints of the arc lie in the angle, (2) each side of the angle contains at least one endpoint of the arc and (3) except for its endpoints, the arc lies in the interior of the angle.

The reason why we talk about the arcs intercepted by angles is that under certain conditions there is a simple relation between the measure of the angle and the measure of the arc.



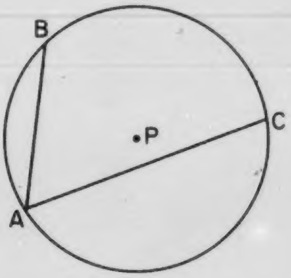
In the figure above we see three inscribed angles, $\angle x$, $\angle y$, $\angle z$, all of which intercept $\widehat{BYC}$ and are inscribed in $\widehat{BXC}$. It looks as if these three angles are congruent. That this is so is a corollary of the following theorem:

THEOREM 12-7. The measure of an inscribed angle is half the measure of its intercepted arc.

Proof:

Given: Circle with center P
A is an inscribed angle intercepting $\widehat{BC}$

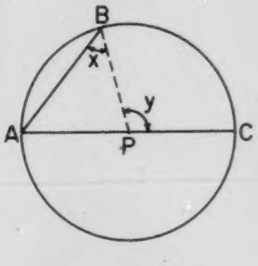
To prove: $m\angle A = \frac{1}{2}m\widehat{BC}$.



There are three possible cases: (1) P is on a side of $\angle A$, say $\overline{AC}$, (2) P is an interior point of $\angle A$, (3) P is an exterior point of $\angle A$.

Case (1). P is on $\overline{AC}$.

Let $\angle x$ and $\angle y$ be as shown. By Theorem 6-10, $m\angle A + m\angle x = m\angle y$. By Theorem 12-2, $\overline{AP} \cong \overline{BP}$ and therefore $m\angle A = m\angle x$. By the substitution property of equality,

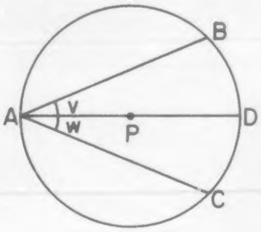


$2m\angle A = m\angle y$. By the definition of the measure of an arc, $m\angle y = m\widehat{BC}$. By the multiplication property of equality we conclude that

$$m\angle A = \frac{1}{2}m\widehat{BC} .$$

Case (2). P is an interior point of A .

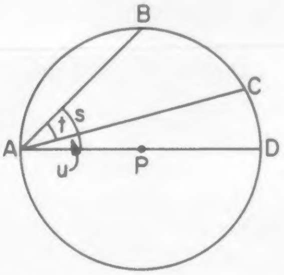
By the Betweenness-Angles Theorem $m\angle A = m\angle v + m\angle w$, and by the Postulate 30, $m\widehat{BDC} = m\widehat{BD} + m\widehat{DC}$.



By Case (1), $m\angle v = \frac{1}{2}m\widehat{BD}$ and $m\angle w = \frac{1}{2}\widehat{DC}$. Therefore

$$m\angle A = \frac{1}{2}(m\widehat{BD} + m\widehat{DC}) = \frac{1}{2}m\widehat{BC} .$$

Case (3). P is an exterior point of $\angle A$.



$$\begin{aligned} m\angle BAC = m\angle t &= m\angle s - m\angle u \\ &= \frac{1}{2}(m\widehat{BD} - m\widehat{CD}) \\ &= \frac{1}{2}m\widehat{BC} . \end{aligned}$$

From this theorem we get two very important corollaries:

Corollary 12-7-1. An angle inscribed in a semicircle is a right angle.

This is so because such an angle intercepts a semicircle, which has measure 180 .

Corollary 12-7-2. Angles inscribed in the same arc are congruent.

This is so because all such angles intercept the same arc.

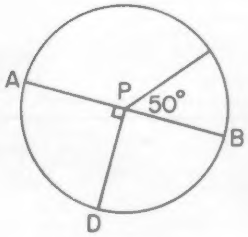
We now say what we mean by congruent arcs. Just as we already did for segments and angles, we state our definition in terms of the appropriate measure.

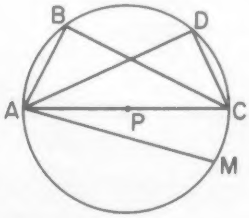
DEFINITION. In the same circle, or in congruent circles, two arcs are called congruent if they have the same measure.

Corollary 12-7-3. Congruent angles inscribed in congruent circles intercept congruent arcs.

Problem Set 12-4a

1. In the circles in each diagram, P is the center and points A, B, C, D, M are contained in the circles as indicated.





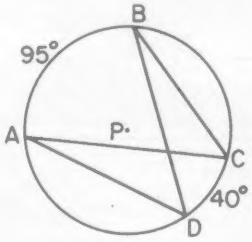


Figure a	Figure b	Figure c
$\overline{AB}$ contains P	$\overline{AC}$ contains P	$m\widehat{AB} = 95$
$\overline{DP} \perp \overline{AB}$		$m\widehat{DC} = 40$

(a) Refer to Figure a.

(1) Name the central angles.
(2) The measures of all central angles are positive numbers less than what number?
(3) Name all the minor arcs in the circle.

(4) Name all the major arcs in the circle.

(5) Name the pairs of congruent arcs in the circle.

(6) Name two arcs which do not have central angles associated with them.

(b) Refer to Figure b.

(1) How many inscribed angles are shown?

(2) Name any 6 of these and for each name the arc in which it is inscribed and the arc it intercepts.

(3) For which of the inscribed angles can you give the degree measure?

(c) Refer to Figure c.

(1) Name the arcs intercepted by $\angle A$, $\angle B$, $\angle C$, $\angle D$.

(2) Give the degree measure of each angle in Part 1.

(3) Name two pairs of congruent angles. Justify your statements in two ways. Write the definition, theorem or corollary used.

2. Consider angles formed by secant-rays, tangent-rays, and/or chord-rays. Call the vertex of such angles V . Make diagrams which indicate all possible pairing of chords, secants, tangents to form these angles if:

(a) V is in the exterior of the circle.

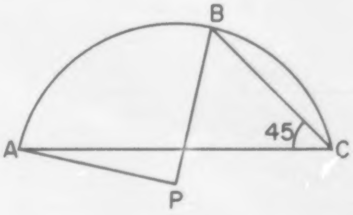
(b) V is on the circle.

(c) V is in the interior of the circle.

3. The center of an arc is the center of the circle of which the arc is a part. How would you find the center of $\widehat{AB}$?

4. Given: P is the center of $\widehat{AC}$, $m \angle C = 45$.

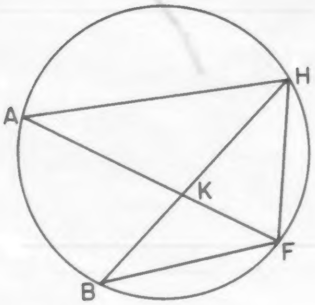
Prove: $\overline{BP} \perp \overline{AP}$.



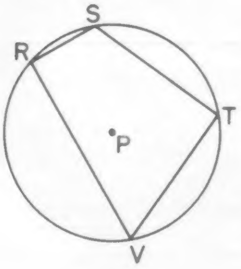
5. If $m\widehat{AB} = m\widehat{BF}$,

   (a) Prove $\triangle AHK \sim \triangle BHF$.

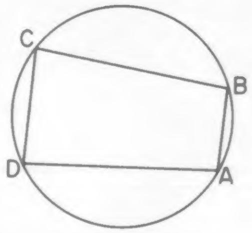
   (b) What other triangle in the figure is similar to $\triangle BHF$?



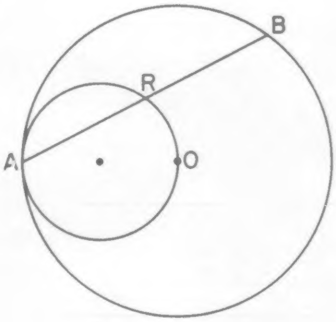
6. In the circle with center P , let $m\angle R = 85$, $m\widehat{RS} = 40$, $m\widehat{TV} = 90$. Find the degree measures of the other arcs and angles indicated in the figure.



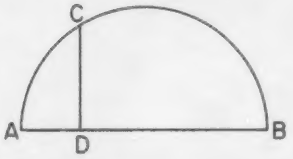
7. An inscribed quadrilateral is a quadrilateral having all of its vertices on a circle. Prove the theorem: The opposite angles of an inscribed quadrilateral are supplementary.



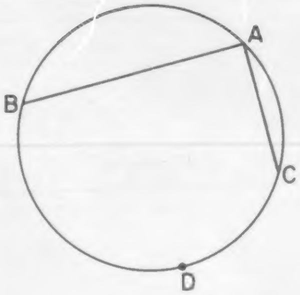
8. The two circles in this figure are tangent at A and the smaller circle passes through O , the center of the larger circle. Prove that any chord of the larger circle with endpoint A is bisected by the smaller circle.



9. In the figure, $\widehat{ACB}$ is a semicircle and $\overline{CD} \perp \overline{AB}$. Prove that $(CD)^2 = AD \cdot DB$. Refer to Corollary 7-7-1.

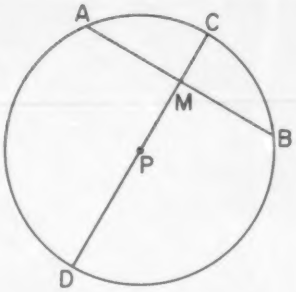


10. Prove the following converse of Corollary 12-7-1: If an angle inscribed in a circular arc is a right angle, then the arc is a semicircle.



11. A, B, C, D are points on a circle such that $\overrightarrow{AC}$ bisects $\angle BAD$. Prove that $m\widehat{DC} = m\widehat{CB}$.

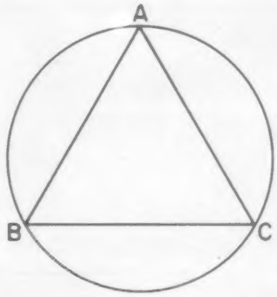
12. Prove: A diameter perpendicular to a chord of a circle bisects both arcs determined by the chord.



13. Prove: If a line bisects both an arc and its chord, it contains the center of the circle.

14. In the diagram $\overline{AB} \cong \overline{AC}$.

Prove that $m\widehat{AB} = m\widehat{AC}$.

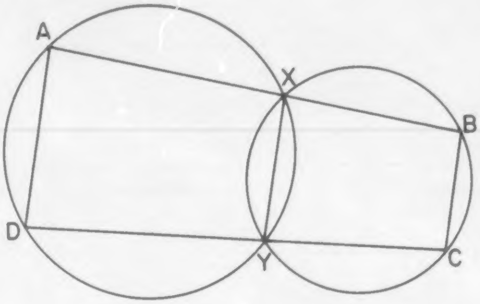


15. Prove Corollary 12-7-1 by using coordinates. [Hint: Let an xy-coordinate system assign (0,0) to the center of the circle. Find the slopes of the rays forming the angle. What must the product of these slopes be?]

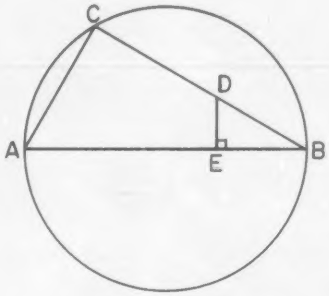
16. $\overline{XY}$ is the common chord of two intersecting circles. $\overline{AB}$ and $\overline{DC}$ are two segments cutting the circles as shown in the figure and containing X and Y respectively.

Prove: $\overline{AD} \parallel \overline{BC}$.

(Hint: See Problem 7.)



17. In this diagram $\overline{AB}$ is a diameter and $\overline{DE} \perp \overline{AB}$.

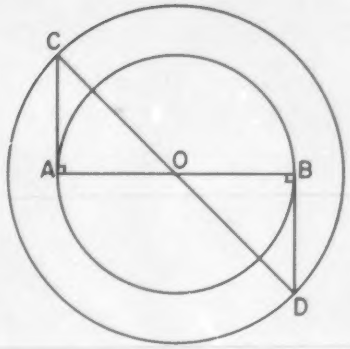


(a) Indicate a correspondence which is a similarity between the triangle with vertices A, C, B and the triangle with vertices B, D, E .

(b) Express the relation between the corresponding sides of the two triangles in (a) using the symbols for a proportionality.

(c) Prove: $BD \cdot BC = BA \cdot BE$.

18. In this figure, $\overline{AB}$ is a diameter of the smaller of two concentric circles, both with center O, and $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are tangent to the smaller circle. $\overline{CO}$ and $\overline{DO}$ are radii of the larger circle.

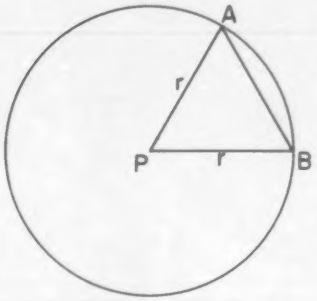


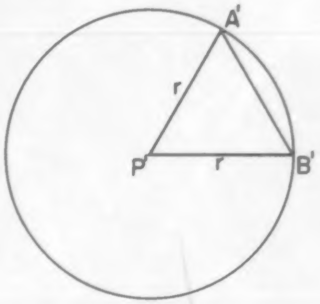
Prove that $\overline{CD}$ is a diameter of the larger circle.

(*Hint*: Consider $\overline{AD}$ and $\overline{CB}$.)

We return to congruent arcs and related chords.

THEOREM 12-8. In the same circle or in congruent circles, if two chords, not diameters, are congruent, then so are the associated minor arcs.





Proof: Referring to the above figure, we need to show that if $AB = A'B'$, then $\overset{\frown}{AB} \cong \overset{\frown}{A'B'}$. By the S.S.S. Postulate we have

$$\triangle APB \cong \triangle A'P'B' .$$

Therefore $\angle P \cong \angle P'$. Since $m\overset{\frown}{AB} = m\angle P$ and $m\overset{\frown}{A'B'} = m\angle P'$, this means that $\overset{\frown}{AB} \cong \overset{\frown}{A'B'}$, which was to be proved.

The converse is also valid and the proof is very similar.

THEOREM 12-9. In the same circle or in congruent circles, if two arcs are congruent, then so are the associated chords.

That is, referring to the figure above, if $\widehat{AB} \cong \widehat{A'B'}$, then $\overline{AB} \cong \overline{A'B'}$. If the major arcs are known to be congruent, then the same conclusion holds.

We now relate measure of other types of angles to measures of intercepted arcs. The figures below show the types of angles we consider.

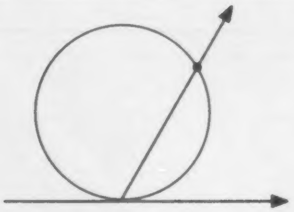


Figure a

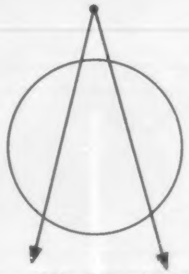


Figure b

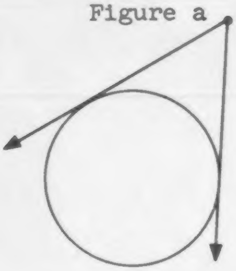


Figure c

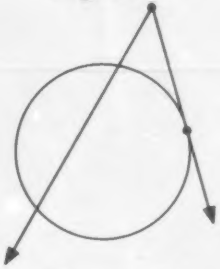


Figure d

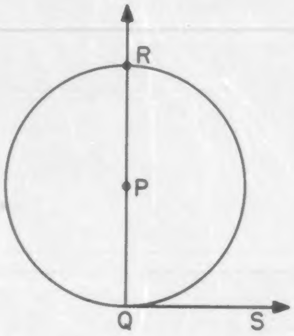
In Figure a the one ray is contained in a tangent and the other ray contains a chord. Describe the rays in Figure b, in Figure c, in Figure d.

DEFINITION. If the vertex of an angle is on a circle and one of its sides is contained in a tangent, and its other side contains a chord, then the angle is called a tangent-chord angle.

DEFINITIONS. If the vertex of an angle is an exterior point of a circle and its sides are contained in two secants, or two tangents, or a secant and a tangent, then it is called respectively a secant-secant angle, or a tangent-tangent angle or a secant-tangent angle.

THEOREM 12-10. The measure of a tangent-chord angle is one-half the measure of its intercepted arc.

Proof:



Given: Circle with center P.
$\overleftrightarrow{QS}$ is a tangent at Q.
$\overline{QR}$ is a chord. $\widehat{QR}$ is the intercepted arc of $\angle RQS$.

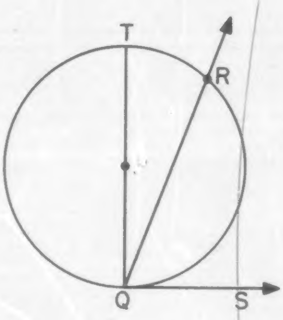
To prove: $m\angle RQS = \frac{1}{2}m\widehat{QR}$.

We consider three cases.

Case (1). P is in $\overrightarrow{QR}$.
Case (2). P is an exterior point of $\angle RQS$.
Case (3). P is an interior point of $\angle RQS$.

Case (1). P is on $\overrightarrow{QR}$. Then by Corollary 12-4-1 $\angle RPQ$ is a right angle. Since $m\widehat{RQ} = 180$, $m\angle RPQ = \frac{1}{2}m\widehat{QR}$.

Case (2). P is an exterior point of $\angle RQS$. Consider diameter $\overline{QT}$. By the Betweenness-Angles Theorem
$m\angle RQS = m\angle TQS - m\angle TQR$.
$m\angle TQS = \frac{1}{2}\cdot 180$, $m\angle TQR = \frac{1}{2}m\widehat{TR}$.
Therefore $m\angle RQS = \frac{1}{2}(180 - m\widehat{TR}) = \frac{1}{2}m\widehat{RS}$.



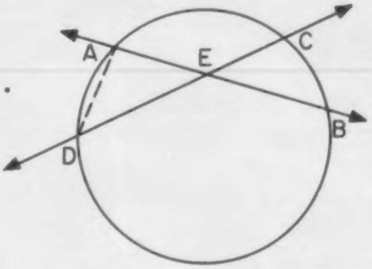
Case (3) is left as a problem.

THEOREM 12-11. The measure of an angle whose vertex is in the interior of a circle and whose sides are contained in two secants, is one-half the sum of the measures of the intercepted arcs.

Proof:

Given: A circle with secants $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersecting at E .

To prove: $m\angle DEB = \frac{1}{2}(m\widehat{DB} + m\widehat{AC})$.

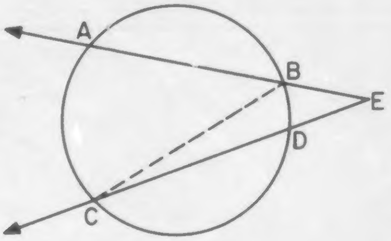


The remainder of the proof is left as a problem.

(Hint: $m\angle DEB = m\angle EAD + m\angle ADE$) .

THEOREM 12-12. The measure of a secant-secant angle, or a tangent-tangent angle or a secant-tangent angle is one-half the difference between the measures of the intercepted arcs.

The proof of this theorem for a secant-secant angle should suggest the proofs for the remaining two angles.



Let the secants be as shown in the diagram.

$m\angle ABC = m\angle E + m\angle BCD$. Why?

Or $m\angle E = m\angle ABC - m\angle BCD$.

The rest of the proof is easily completed by noting that

$m\angle ABC = \frac{1}{2}\widehat{AC}$ (Why?) and

$m\angle BCD = \frac{1}{2}m\widehat{BD}$.

Problem Set 12-4b

1. Consider the points in the following diagrams to be located as the figures suggest. The degree measures indicated are assigned to the arcs.

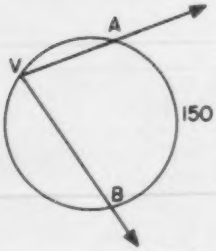


Figure (1)

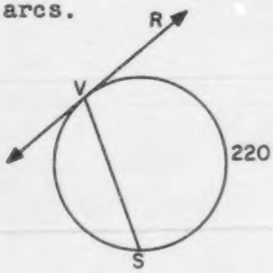


Figure (2)

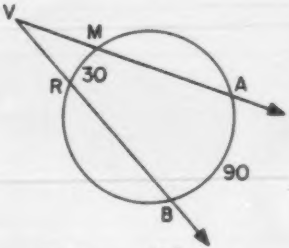


Figure (3)

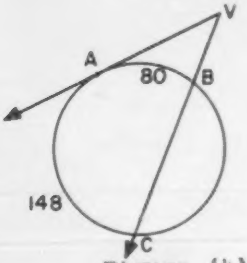


Figure (4)

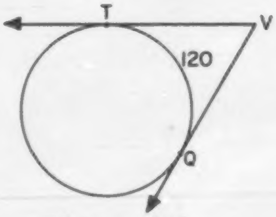


Figure (5)

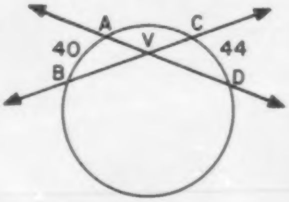


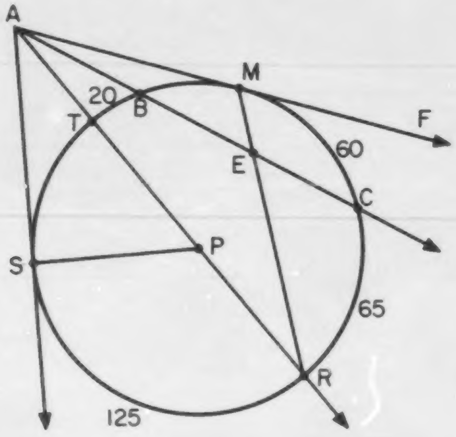
Figure (6)

(a) Match the numbers (1) through (5) from the diagrams with the appropriate angle-name selected from tangent-tangent angle, tangent-secant angle, secant-secant angle, tangent-chord angle, central angle, inscribed angle.

(b) Give the measures of each indicated angle.

(1) Figure (1), m $\angle AVB$ =
(2) Figure (2), m $\angle RVS$ =
(3) Figure (3), m $\angle AVB$ =
(4) Figure (4), m $\angle AVB$ =
(5) Figure (5), m $\angle TVQ$ =
(6) Figure (6), m $\angle AVB$ =

2. Find the measures of each indicated part. The arcs have degree measures as marked.



P is the center of the circle. E is in the interior of the circle. F is in the exterior of the circle. S, T, B, M, C, R are on the circle. $\overrightarrow{AS}$ and $\overrightarrow{AM}$ are tangents to the circle at S and M respectively. $\overline{TR}$ is a diameter.

(a)	$m\widehat{BM}$	(f)	$m\angle ASP$
(b)	$m\widehat{ST}$	(g)	$m\angle TRM$
(c)	$m\angle FMR$	(h)	$m\angle SAR$
(d)	$m\angle FAC$	(i)	$m\angle SPR$
(e)	$m\angle CAR$	(j)	$m\angle CER$

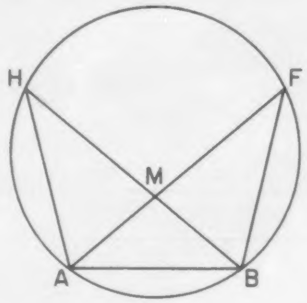
3. Prove Theorem 12-9: In the same circle or in congruent circles, if two arcs are congruent, then so are the associated chords.

4. In the figure AF = BH .

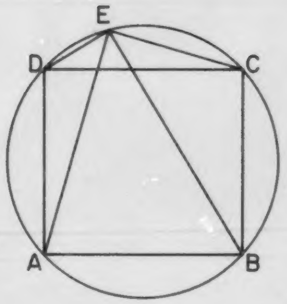
Prove:

(a) $\widehat{FB} \cong \widehat{AH}$

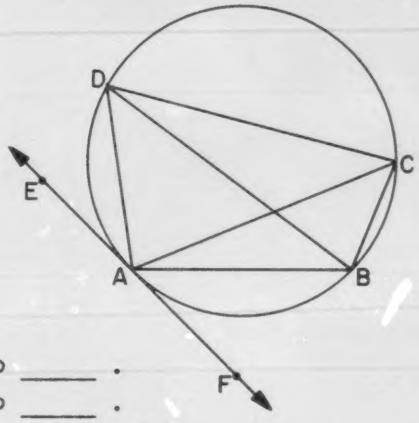
(b) $\triangle BMF \cong \triangle AMH$



5. ABCD is a square. E is any point of $\widehat{DC}$, as shown in this figure. Prove that $\overrightarrow{EA}$ and $\overrightarrow{EB}$ trisect $\angle DEC$.

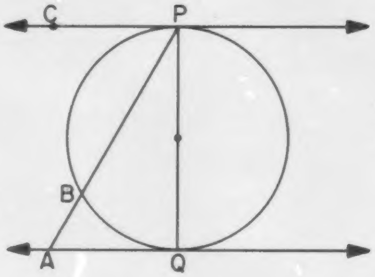


6. In the figure, A, B, C, D are on the circle and $\overleftrightarrow{EF}$ is tangent to the circle at A . Complete the following statements:

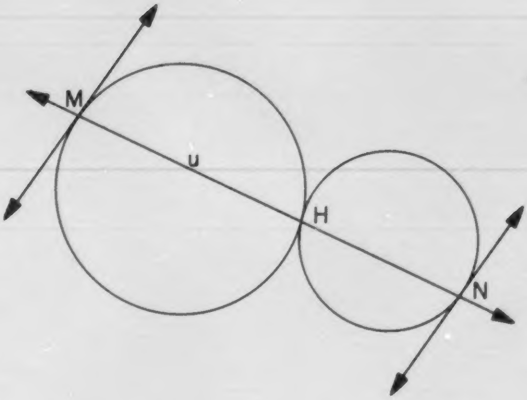


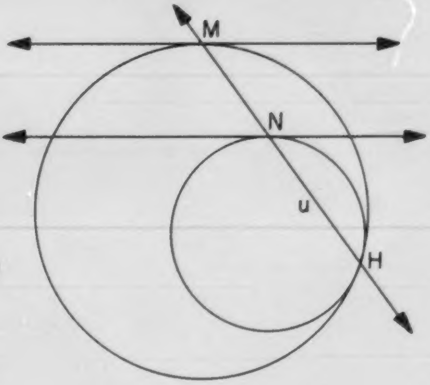
(a) $\angle BDC \cong$ _____ .
(b) $\angle ADC \cong$ _____ .
(c) $\angle ACB \cong$ _____ $\cong$ _____ .
(d) $\angle EAD$ is supplementary to _____ .
(e) $\angle DAB$ is supplementary to _____ .
(f) $\angle ABC$ is supplementary to _____ .
(g) $\angle DAE \cong$ _____ $\cong$ _____ .
(h) $\angle DBA$ is supplementary to _____ .
(i) $\angle ADB$ is supplementary to _____ .
(j) $\angle DAC \cong$ _____ .

7. In the figure $\overleftrightarrow{CP}$ and $\overleftrightarrow{AQ}$ are tangents, $\overline{PQ}$ is a diameter of the circle. If $m\widehat{PB} = 120$ and if the radius of the circle is 3 , find the length of $\overline{AP}$.



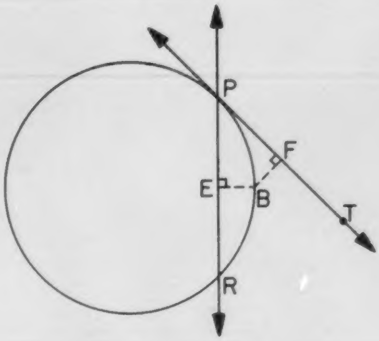
8. Two circles are tangent, either internally or externally, at a point H . Let u be any line through H meeting the circles again at M and N . Prove that the tangents at M and N are parallel.



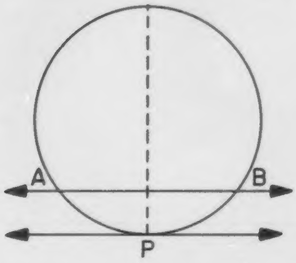


9. Given: Tangent $\overleftrightarrow{PT}$ and secant $\overleftrightarrow{PR}$. B is the midpoint of $\overset{\frown}{PR}$.

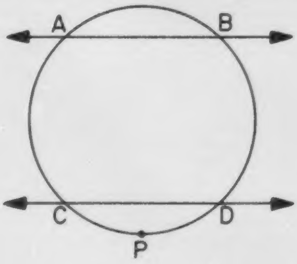
Prove: B is equidistant from $\overleftrightarrow{PT}$ and $\overleftrightarrow{PR}$.



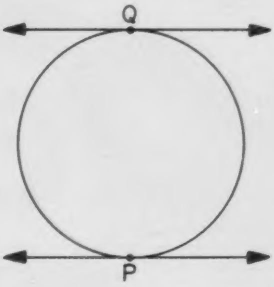
10. Prove the theorem: If two parallel lines intersect a circle, they intercept congruent arcs.



Case I
(One tangent, one secant)



Case II
(Two secants)



Case III
(Two tangents)

*11. Consider the circle O , $\{(x,y): x^2 + y^2 = 25\}$, and the point $Q(0,3)$.

(a) Find the intersections A,B of the line $\overleftrightarrow{QA}$ with the circle O , given $\overleftrightarrow{QA} = \{(x,y): y = 3\}$.

(b) Find the intersections C,D of the line $\overleftrightarrow{QC} = \{(x,y): x = 0\}$, with the circle O .

(c) What is the product of the lengths of segments $\overline{QA}$ and $\overline{QB}$? Of segments $\overline{QC}$ and $\overline{QD}$? Are these products equal? (If not, check your work.)

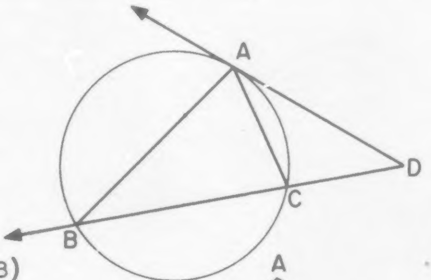
*12. Consider the circle $O = \{(x,y): x^2 + y^2 = 25\}$ and the point $P(8,3)$.

(a) Find the points A,B which are the intersections of $\overleftrightarrow{PA}$ which equals $\{(x,y): y = 3\}$, with the circle O .

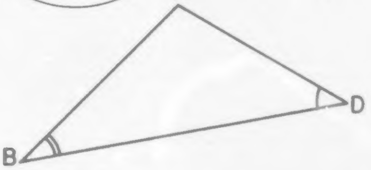
(b) Find the points C,D which are the intersections of $\overleftrightarrow{PC} = \{(x,y): y = x - 5\}$ with the circle O .

(c) Find the product $PA \cdot PB$ and the product $PC \cdot PD$. (If these products are not equal, check your work.)

*13. Given $\overleftrightarrow{AD}$ tangent to the circle at A and the secant $\overleftrightarrow{BD}$ intersecting the circle at B and C .

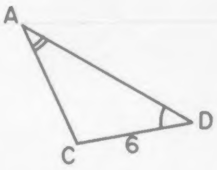


(a) What is the relation between $\triangle ADB$ and $\triangle CDA$? Why?



(b) Why does AD (of $\triangle ADB$) $= k \cdot CD$ (of $\triangle CDA$) ? Why does BD (of $\triangle ADB$) $= k \cdot AD$ (of $\triangle CDA$) ?

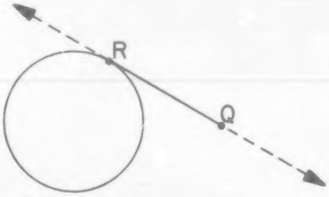
(c) Assume $CD = 6$. Express AD in terms of k . Express BD in terms of k .



(d) Compare $AD \cdot AD$ with $BD \cdot CD$. Would this relation be true for all values of k ? Would it be true for every value assigned to CD ?

12-5. Lengths of Tangent and Secant Segments.

DEFINITION. If the line $\overleftrightarrow{QR}$ is tangent to a circle at R, then the segment $\overline{QR}$ is a tangent-segment from Q to the circle.

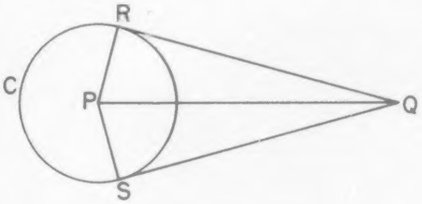


THEOREM 12-13. The two tangent-segments to a circle from an external point are congruent, and form congruent angles with the line joining the external point to the center of the circle.

Proof:

Given: $\overline{QR}$ is tangent to the circle C at R, and $\overline{QS}$ is tangent to C at S.

To prove: $\overline{QR} \cong \overline{QS}$
$\angle PQR \cong \angle PQS$

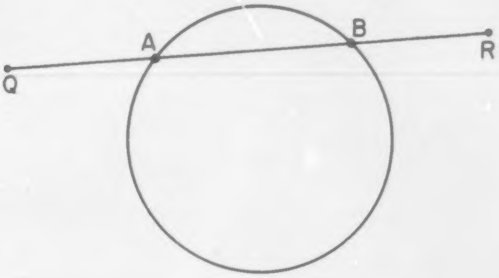


By Corollary 12-4-1, $\triangle PQR$ and $\triangle PQS$ are right triangles, with right angles at R and S. Obviously $\overline{PQ} \cong \overline{PQ}$, and $\overline{PR} \cong \overline{PS}$ because R and S are points of the circle. By the Hypotenuse-Leg Theorem, this means that

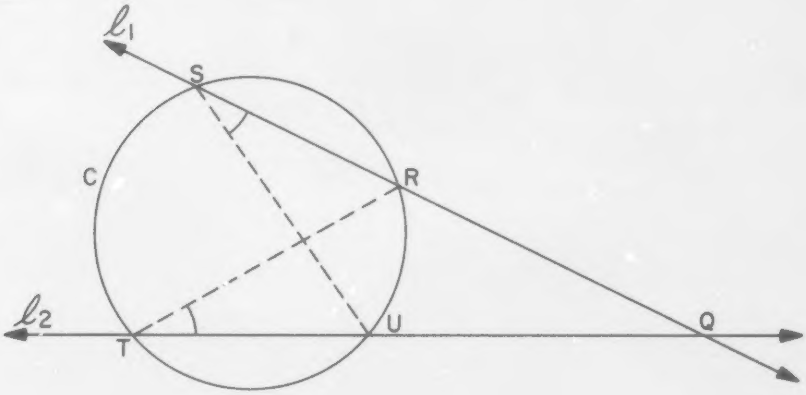
$$\triangle PQR \cong \triangle PQS .$$

Therefore $\overline{QR} \cong \overline{QS}$, and $\angle PQR \cong \angle PQS$, which was to be proved.

DEFINITION. If secant $\overleftrightarrow{QR}$ intersects a circle in A and B such that A is between Q and B, then $\overline{QB}$ is called the secant-segment from Q to the circle and $\overline{QA}$ is called the external secant-segment from Q to the circle.



THEOREM 12-14. The product of the length of a secant-segment from a given exterior point and the length of its external secant-segment is constant for any secant containing the given point.



Proof: Let Q be the given point and let $\overline{QS}$ and $\overline{QT}$ be two secant-segments, having respective external secant-segments $\overline{QR}$ and $\overline{QU}$. By the A.A. Theorem for similar triangles we prove

$$\triangle SQU \sim \triangle TQR .$$

It follows that

$$(QS\ ,\ QT) \underset{p}{=} (QU\ ,\ QR)$$

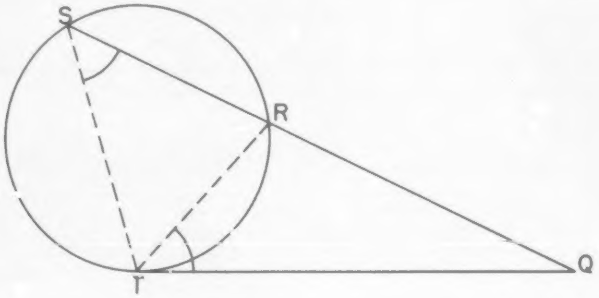
and therefore

$$QS \cdot QR = QU \cdot QT\ .$$

We prove that the product $QS \cdot QR$ is equal to the product of the length of any secant-segment from Q and the length of its external secant-segment. This proves the constant to be $QS \cdot QR$.

Notice that this theorem means that the product $QR \cdot QS$ is determined merely by the given circle and the given external point, and is independent of the choice of the secant. (The theorem tells us that any other secant gives the same product.) This constant product is called the power of the point with respect to the circle.

The following theorem asserts that, in the figure below, $QR \cdot QS = (QT)^2$.



THEOREM 12-15. Given a tangent-segment $\overline{QT}$ to a circle at T and a secant through Q , intersecting the circle in points R and S . Then

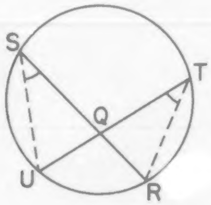
$$QR \cdot QS = (QT)^2\ .$$

The main steps in the proof are as follows. You should find the reasons in each case.

1. $m\,\angle S = \frac{1}{2} m\widehat{TR}$.
2. $m\,\angle RTQ = \frac{1}{2} m\widehat{TR}$.
3. $\angle S \cong \angle RTQ$.
4. $\triangle QRT \sim \triangle QTS$.
5. $(QR,QT) \underset{p}{=} (QT,QS)$.
6. $QR \cdot QS = (QT)^2$.

The following theorem is a further variation on the preceding two; the difference is that now we are going to draw two lines through a point in the interior of the circle. The theorem says that in the figure below, we have

$$QR \cdot QS = QU \cdot QT .$$



You will recognize this theorem as the generalization arrived at in Problem 5 of Problem Set 1-4.

THEOREM 12-16. If two chords of a circle intersect, the product of the lengths of the segments of one is equal to the product of the lengths of the segments of the other.

The main steps in the proof are as follows. You should find the reason in each case.

1. $\angle S \cong \angle T$.
2. $\angle SQU \cong \angle TQR$.
3. $\triangle SQU \sim \triangle TQR$.
4. $(QS,QT) \underset{p}{=} (QU,QR)$.
5. $QR \cdot QS = QU \cdot QT$.

Problem Set 12-5

1. Complete the following statements by replacing the blanks with appropriate words or expressions.

   (a) If M is any point in the exterior of a circle, there are ________ tangent-segments to the circle from M and their _______ are equal. If P is the center of the circle, then $\overrightarrow{MP}$ is the ________ of the ________ containing the tangent-segments. Illustrate with a diagram.

(b) If $\overline{RS}$ is a secant-segment, R is in the exterior of a circle and S is ________ the circle. If $\overline{RA}$ is the external secant-segment and a subset of $\overline{RS}$, then A is _____the circle and ________ is between ________ and ________ . Illustrate with a diagram.

(c) If in a circle three chords $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$ intersect at X and if $AX \cdot XB = 12$, then what is $CX \cdot XD$? What is $EX \cdot XF$?

2. Points P, R, M are in the exterior of the circles and A, B, X, H, K, S, T are on the circles as indicated in the diagrams. R is the center of the circle in Figure a. A, B, and X are points of contact of tangents in Figures (a) and (b), respectively.

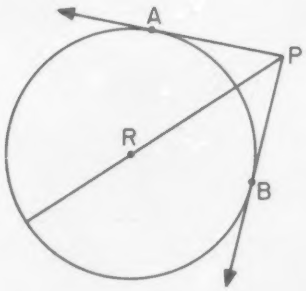


Figure a.

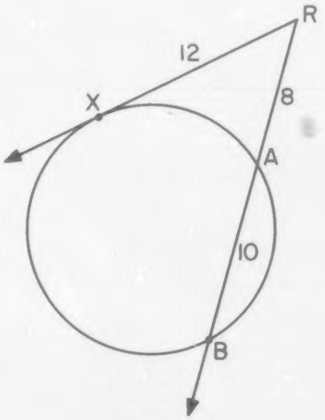


Figure b.

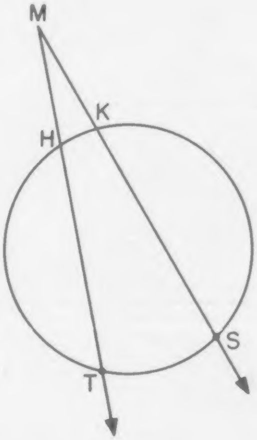


Figure c.

(a) Refer to Figure a.

(1) Segments $\overline{PA}$ and $\overline{PB}$ are ________ .

(2) If PA = 10 , then PB = ________ . Why? (State the theorem.)

(3) If m $\angle APR = 45$, then m $\angle BPR$ = ________ . Why? (State the theorem.)

(b) Refer to Figure b.

(1) The tangent-segment is _____ .

(2) The secant-segment is ______ .

(3) The external secant-segment is _____ .

(4) RA + AB = _____ . What theorem did you use?

(5) If XR = 12 , could RA = 8 when AB = 10 ? Justify your answer.

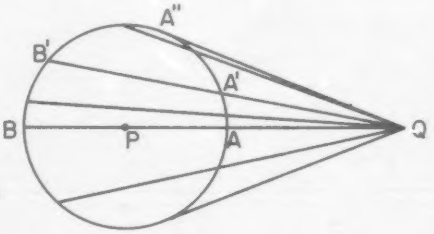
(6) If XR = 12 , could RA = 6 when AB = 18 ? Given another pair of numbers which could be the measures of $\overline{RA}$ and $\overline{RB}$ respectively.

(c) Refer to Figure c.

(1) If MH = a , HT = b , MK = x , KS = y then a · (_____) = x · (_____) . State the theorem which justifies your answer.

(2) Could MH = 3 , HT = 17 , MK = 4 , KS = 11 ? Explain.

3. In the diagram P is the center of the circle, Q is a point in the exterior of the circle and A and B are on the circle. $\overline{QB}$ is a secant-segment, $\overline{QA}$ is an external secant-segment and $\overline{QA''}$ is a tangent-segment.



We want to discover the relation between Theorem 12-14 and Theorem 12-15 by noting relations in the diagram.

(a) In what position does $\overline{QB}$ appear to have its greatest length?

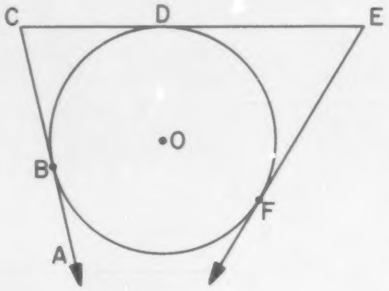
(b) In what position does the external secant-segment appear the shortest?

(c) If $\overline{QB}$ takes on a sequence of positions on the circle, changing from position $\overline{QPB}$ to position $\overline{QA''}$, the length of $\overline{QB}$ appears to ________ . The length of the external secant-segment appears to ________ . The length of the secant-segment appears to decrease and approach the length of the ________ $\overline{QA''}$. The length of the external secant-segment appears to increase and approach the length of ________ ________.

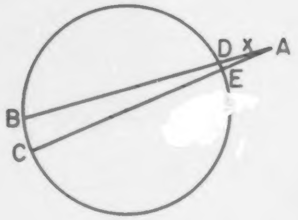
(d) Since $QB \cdot QA = QB' \cdot QA'$, etc. for all positions of B' and A' , when B' becomes A'' and A' becomes A'' we would expect $QB \cdot QA$ to equal $(Q\_\_\_) \cdot (Q\_\_\_)$. Thus the situation in Theorem 12-15 is what we might call the limiting case of the situation in Theorem 12-14.

4. $\overleftrightarrow{AC}$, $\overleftrightarrow{CE}$ and $\overleftrightarrow{EF}$ are tangent to a circle at B, D, and F respectively.

Prove: CB + EF = CE .

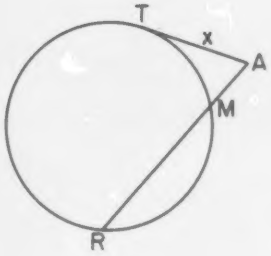


5. Use the data as it appears in the diagrams.



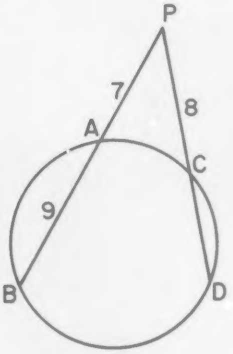
BA = 20
CE = 16
EA = 6

Figure (a)



RA = 16
MA = 4

Figure (b)

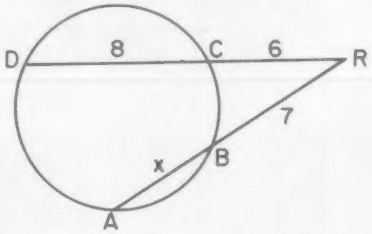


BA = 9
AP = 7
PC = 8

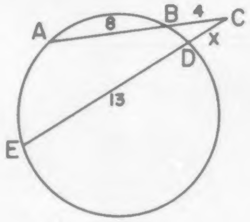
Figure (c)

(a) Use Figure (a) and compute DA .
(b) Use Figure (b) and compute AT .
(c) Use Figure (c) and compute PD .

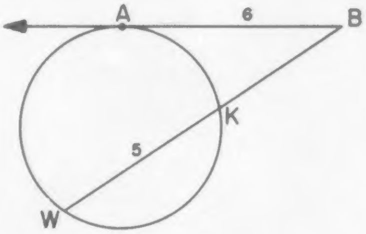
6. In the figure DC = 8 , CR = 6 , RB = 7 . Find BA .



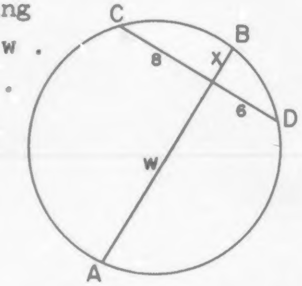
7. Secants $\overleftrightarrow{CA}$ and $\overleftrightarrow{CE}$ intersect the circle at A, B, and D, E as shown in this figure. If AB = 8 , BC = 4 , ED = 13 , find DC .



8. In this figure $\overleftrightarrow{AB}$ is tangent to the circle at A and secant $\overleftrightarrow{BW}$ intersects the circle at K and W . If AB = 6 and WK = 5 , how long is $\overline{BK}$?

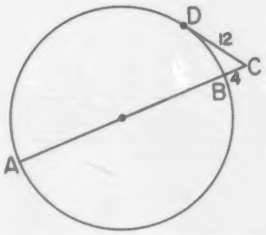


9. Given a circle with intersecting chords as shown and with $x < w$. If AB = 19 , find x and w .



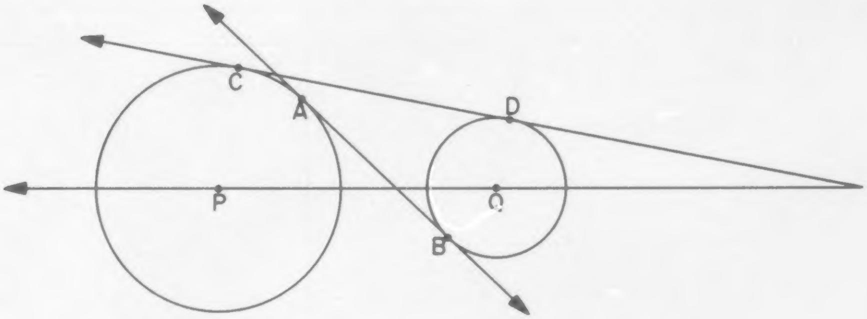
10. Given a circle with a chord of length 12 whose distance from the center is 8, find the radius of the circle.

11. In the figure, $\overline{CD}$ is a tangent-segment to the circle at D and $\overline{AC}$ is a secant-segment which contains the center of the circle. If CD = 12 and CB = 4 , find the radius of the circle.



12. If two tangent-segments to a circle form an equilateral triangle with the chord having the points of tangency as its endpoints, find the measure of each arc of the chord.

13. If a common tangent of two circles meets the line of centers at a point between the centers it is called a common internal tangent. If it does not meet the line of centers at a point between the centers it is called a common external tangent.



In the figure $\overleftrightarrow{AB}$ is a common internal tangent and $\overleftrightarrow{CD}$ is a common external tangent.

(a) In the figure above, how many common tangents are possible? Specify how many of each kind.

(b) If the circles were externally tangent, how many tangents of each kind?

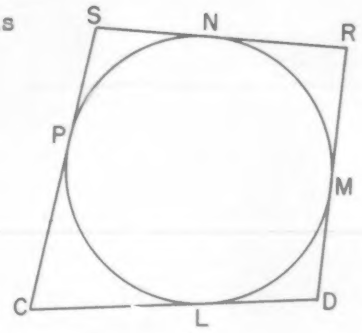
(c) If the circles were intersecting at two points?

(d) If the circles were internally tangent?

(e) If the circles were concentric?

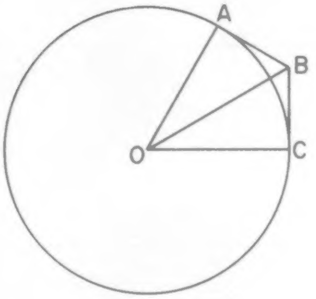
14. Given: The sides of quadrilateral CDRS are tangent to a circle as in the figure.

    Prove: SR + CD = SC + RD .



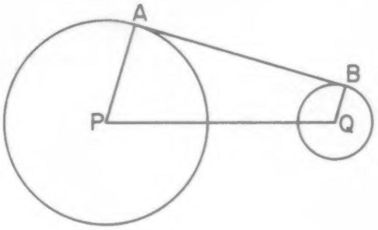
15. $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are tangent to a circle with center O at A and C , respectively, and $m\angle ABC$ equals 120 .

    Prove that AB + BC = OB .



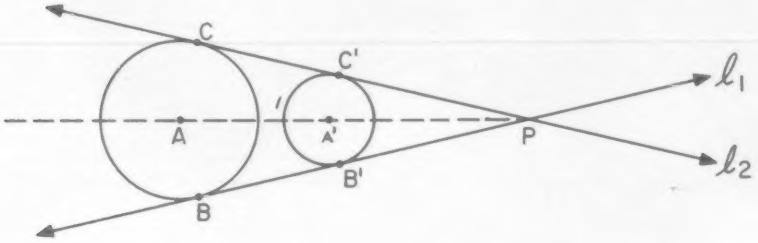
16. The radii of two circles have lengths 22 and 8 respectively, and the distance between their centers is 50 . Find the length of the common external tangent-segment.

    (*Hint*: Draw a perpendicular from Q to $\overline{AP}$.)



17. Two circles have a common external tangent-segment 36 inches long. Their radii are 6 inches and 21 inches respectively. Find the distance between their centers.

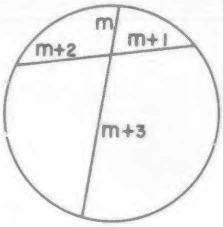
18. In the accompanying diagram ℓ_1 and ℓ_2 are tangent to a circle with center A at B and C respectively. A second circle with center A' lies in the union of $\angle CPB$ and its interior. ℓ_1 and ℓ_2 are tangent to circle A' at B' and C' respectively.



(a) Are $\overrightarrow{PA}$ and $\overrightarrow{PA'}$ distinct? Explain.

(b) If minor arc $\overset{\frown}{BC}$ has a degree measure of 130 , what is the degree measure of minor arc $\overset{\frown}{B'C'}$? Justify your answer.

19. Show that it is not possible for the lengths of the segments of two intersecting chords to be four consecutive integers.



20. Consider circle $C = \{(x,y): (x - 1)^2 + (y + 3)^2 = 64\}$ and lines ℓ , ℓ' such that

$$\ell = \{(x,y): y = 5\} \text{ and } \ell' = \{(x,y): x - y = 12\} .$$

(a) Find the coordinates of P , the intersection of ℓ and ℓ' .

(b) Find the coordinates of T , a point of intersection of ℓ with the circle. (There is only one point in this case; ℓ is tangent to C .)

(c) Find the coordinates of R and S' , two points of intersection of ℓ' (a secant) with the circle.

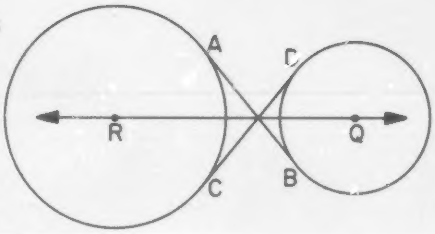
(d) Find PT and square it.

(e) Find PR and PS' and their product.

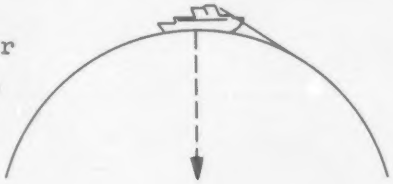
(f) Did you expect $(PT)^2$ and $PR \cdot PS'$ to be equal? Why?

21. Prove: The common internal tangents of two circles meet the line of centers at the same point.

(Hint: Use an indirect proof.)



22. Standing on the bridge of a large ship on the ocean, the captain asked a young officer to determine the distance to the horizon. The young officer took a pencil and paper and in a few moments came up with an answer. On the paper he had written the formula



$$d = \frac{5}{4}\sqrt{h} \text{ miles.}$$

Show that this formula is approximately correct if h is the height in feet of the observer above the water and if d is the distance in miles to the horizon. (Assume the diameter of the earth to be 8,000 miles.)

Review Problems

(Chapter 12, Sections 1-5)

1. Consider the problems below with reference to the four sets C, L_1, L_2 and L_3.

$C = \{(x,y): x^2 + y^2 = 100\}$;

$L_1 = \{(x,y): x = -10\}$;

$L_2 = \{(x,y): y = 6\}$;

$L_3 = \{(x,y): y = \frac{4}{3}x\}$.

(a) C is a __________ with radius __________ and center at the point with coordinates __________.

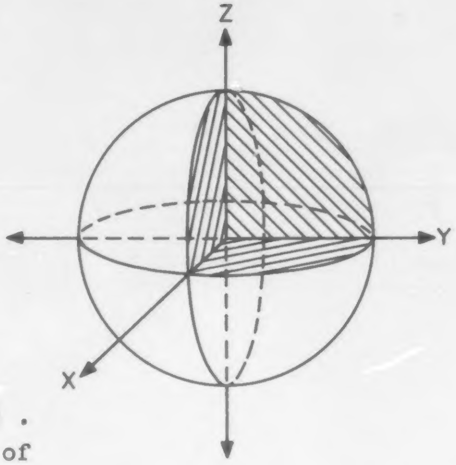
(b) A(6, 8), B(7, -7), and C(-8, 10) are three points in the xy-plane. For each point determine whether it is on the circle, in the interior of the circle or in the exterior of the circle. Show your computations.

(c) Find the intersection of L_1 and C .

(d) Find the intersection of L_2 and C .

(e) Find the intersection of L_3 and C .

2. Consider a sphere S with radius 10 and an xyz-coordinate system which has its origin at the center of S .



(a) Write an equation of S .

(b) Give the coordinates of the points of intersection of S with

  (1) the x-axis
  (2) the y-axis
  (3) the z-axis

(c) Give the intersection of S with the xy-plane, that is, with $\{(x,y,z): z = 0\}$.

(d) Give the intersection of S with the xz-plane.

(e) Give the intersection of S with the yz-plane.

(f) Given the points $A(3,-4,5\sqrt{3})$, $B(3,-5,7)$, $C(9,6,1)$. For each point determine whether it is (1) in S , (2) in the interior of S or (3) in the exterior of S .

3. (a) Write an equation of a circle in the xy-plane which has its center at $(3,-2)$ and its radius equal to 4

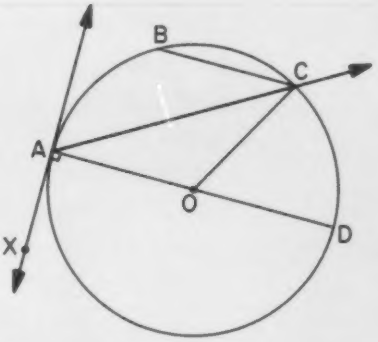
(b) Write an equation of a sphere with center at $(2,-1,3)$ and with radius equal to 3 .

4. From the presentation in Sections 12-1 through 12-5, we have several situations in which two angles, two segments or two arcs are congruent.

   (a) Give 6 conditions under which 2 segments related to a circle are congruent.
   (b) Give 3 circumstances under which an angle related to a circle is a right angle.
   (c) Give 4 conditions under which two angles related to a circle are congruent.
   (d) Give 4 conditions under which 2 arcs have the same degree measure.

5. (a) How is the degree measure of the arc in which an angle is inscribed related to the degree measure of the arc which it intercepts?
   (b) Explain how the relation between the measure of a central angle and the degree measure of its associated arc might be considered a special case of the relation between the measure of an angle formed by two chords which intersect in the interior of a circle and the degree measure of its associated arcs.

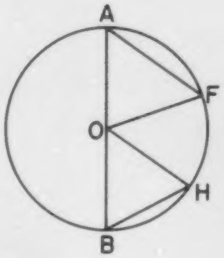
6. For the circle centered at O ,



   (a) $\overline{BC}$ is a ________ .
   (b) $\overline{AD}$ is a ________ .
   (c) $\overleftrightarrow{AC}$ is a ________ .
   (d) $\overline{OA}$ is a ________ .
   (e) $\overleftrightarrow{AX}$ is a ________ .
   (f) $\overset{\frown}{CD}$ is a ________ .
   (g) $\overset{\frown}{ADC}$ is a ________ .
   (h) $\angle BCA$ is an ______ ______ .
   (i) $\angle COD$ is a ______ ______ .

7. Given: In the figure, the circle with center O has diameter $\overline{AB}$. $\overline{AF} \parallel \overline{OH}$, $m\angle A = 55$.

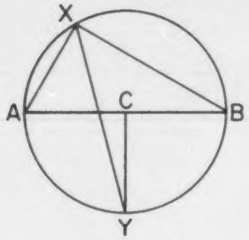
   Find $m\overset{\frown}{BH}$ and $m\overset{\frown}{AF}$.



8. Given: $\overline{AB}$ is a diameter of the circle with center C. $\overrightarrow{XY}$ bisects $\angle AXB$.

Prove: $\overline{CY} \perp \overline{AB}$.

(Hint: Find $m\angle AXY$.)



9. Indicate whether each of the following statements is true or false.

(a) If a point is the midpoint of two chords of a circle, then the point is the center of the circle.

(b) If the measure of one arc of a circle is twice the measure of a second arc, then the chord of the first arc is less than twice as long as the chord of the second arc.

(c) A line which bisects two chords of a circle is perpendicular to each of the chords.

(d) If the vertices of a quadrilateral are on a circle, then each two of its opposite angles are supplementary

(e) If each of two circles is tangent to a third circle, then the two circles are tangent to each other.

(f) A circle cannot contain three collinear points.

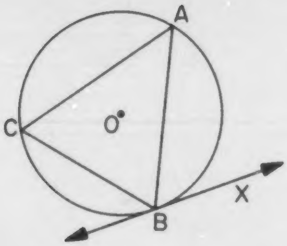
(g) If a line bisects a chord of a circle, then it bisects the minor arc of that chord.

(h) If $\overline{PR}$ is a diameter of a circle and Q is any point in the interior of the circle not on $\overline{PR}$, then $\angle PQR$ is obtuse.

(i) A tangent to a circle at the midpoint of an arc is parallel to the chord of that arc.

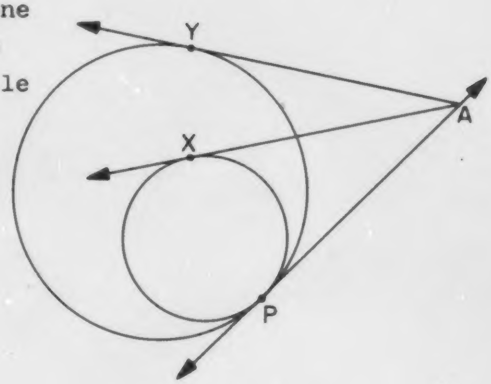
(j) It is possible for two tangents to the same circle to be perpendicular to each other.

10. Given: In the figure $\overleftrightarrow{BX}$ is tangent to the circle at B . AB = AC . $m\overset{\frown}{CB} = 100$. Find $m\angle C$, $m\angle ABX$, and $m\angle CBA$.



11. Given: Two circles tangent at P with common tangent $\overleftrightarrow{AP}$. $\overleftrightarrow{AX}$ is tangent to one circle at X and $\overleftrightarrow{AY}$ is tangent to the other circle at Y .

Prove: AY = AX .



12. A hole 40 inches in diameter is cut in a sheet of plywood, and a globe 50 inches in diameter is set in this hole. How far below the surface of the board will the globe sink?

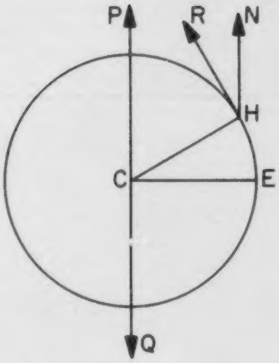
13. A wheel is broken so that only a portion of the rim remains. In order to find the diameter of the wheel the following measurements are made: three points C, A, and B are taken on the rim so that chord $\overline{AB} \cong$ chord $\overline{AC}$. The chords $\overline{AB}$ and $\overline{AC}$ are each 15 inches long, and the chord $\overline{BC}$ is 24 inches long. Find the diameter of the wheel.

14. Diameter $\overline{AD}$ of a circle with center C contains a point B which lies between A and C . Prove that $\overline{BA}$ is the shortest segment joining B to the circle and $\overline{BD}$ is the longest.

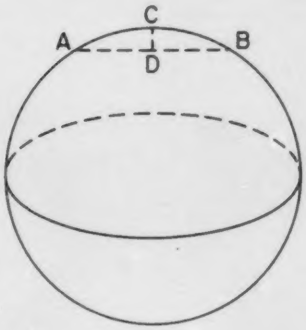
15. Given: Circle with center C, $\overline{EC} \perp \overleftrightarrow{PQ}$, $\overrightarrow{HN} \parallel \overleftrightarrow{PQ}$, and $\overrightarrow{HR}$ tangent to the circle at H.

Prove: $m\widehat{HE} = m\angle RHN$.

(Note: The circle may be considered to represent the earth, with $\overleftrightarrow{PQ}$ the earth's axis, $\angle RHN$ the angle of elevation of the North Star, and $m\widehat{HE}$ the latitude of a point H.)



16. Assume that the earth is a sphere of radius 4,000 miles. A straight tunnel $\overline{AB}$ 200 miles long connects two points A and B on the surface, and a ventilation shaft $\overline{CD}$ is constructed at the center of the tunnel. What is the length, in miles, of this shaft?



17. Describe the sets indicated below.

(a) $T = \{(x,z): x^2 + z^2 > 4\}$.

(b) $M = \{(x,y): (x - 2)^2 + (y + 4)^2 = 49\}$.

(c) $N = \{(y,z): y^2 + z^2 < 9\}$.

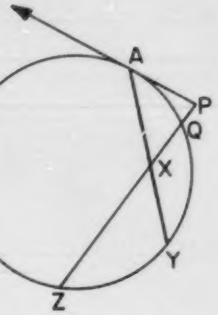
(d) $R = \{(x,y,z): x^2 + y^2 + z^2 = 25, z = 3\}$.

(e) The intersection of $A = \{(x,y,z): x^2 + y^2 + z^2 = 1\}$ and $B = \{(x,y,z): |x| = 1\}$.

(f) The intersection of $D = \{(x,y,z): x^2 + y^2 + z^2 = 16\}$ and $F = \{(x,y,z): x^2 + y^2 + z^2 = 8\}$.

(g) The intersections of $T = \{(x,y,z): x^2 + y^2 + z^2 = 25\}$ and $U = \{(x,y,z): x^2 + y^2 = 9\}$.

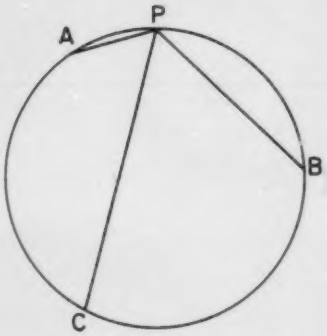
18. In the figure, $\overleftrightarrow{AP}$ is tangent to the circle at A . AP = PX = XY . If PQ = 1 and QZ = 8 , find AX .



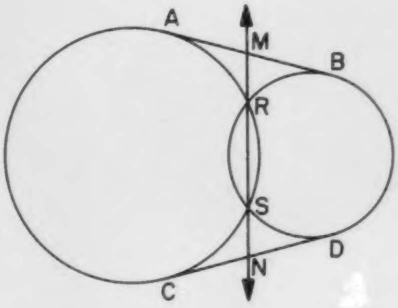
19. Given: $\widehat{AB}$, $\widehat{BC}$ and $\widehat{CA}$ are 120° arcs on a circle and P is a point on $\widehat{AB}$.

Prove: PA + PB = PC .

(*Hint:* Consider a parallel to $\overline{PB}$ through A intersecting $\overline{PC}$ in R and the circle in Q .)

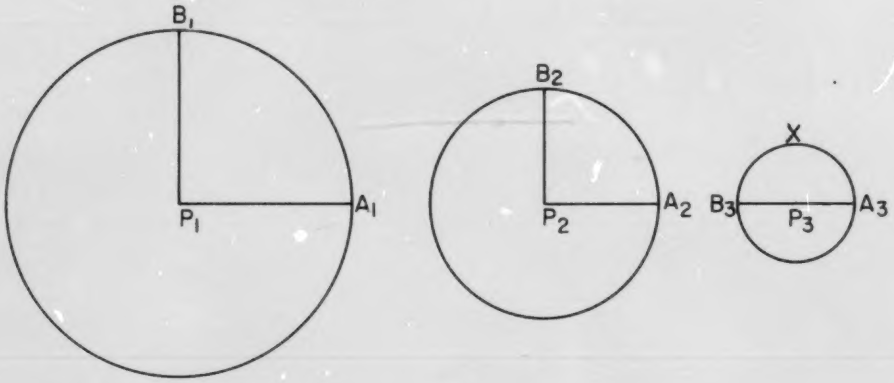


20. Prove that if two circles intersect, the common secant bisects both common tangent-segments.



12-6. The Circumference of a Circle; the Number π.

It makes sense to ask of someone who made a trip how far he went. If he traveled in a straight line the answer would be the distance between his starting point and arrival point. If he traveled in a curved path the answer would not be so easy to give. If the path were a circular arc, the degree measure of the arc is not a satisfactory way of describing its length. Can you see that it is possible for two arcs to have the same degree measure and have different lengths? Can you also see that it is possible for two arcs to have different degree measures and have the same length?



$m\widehat{A_1B_1} = 90$	$m\widehat{A_2B_2} = 90$	$m\widehat{A_3XB_3} = 180$
length of $\widehat{A_1B_1}$	length of	length of
(in mm.) = 93	$\widehat{A_2B_2} = 62$	$m\widehat{A_3XB_3} = 62$

We are going to try to say what we mean by the length of circular arcs and to derive ways of finding such lengths. The subject is discussed more thoroughly in a branch of mathematics known as "calculus," where all kinds of curved paths are discussed. We first proceed informally, referring to the physical world. We imagine that we have made a complete circuit around a circular path and inquire how far we have gone. We call this distance the circumference of the circle and denote it by C . It seems reasonable to suppose that if we want to measure C approximately, we can do it by inscribing a regular polygon with a large number of sides and then measuring the perimeter of the polygon. That is, the perimeter p ought to be a good approximation to C when n , the number of sides, is large. Putting it another way, if we decide how close we want p to be to C , we ought to be able to get p to be this close to C merely by making n large enough. We describe this situation in symbols by writing $p \longrightarrow C$, and we say that p has C as a limit.

We cannot prove this, however; and the reason why we cannot prove it is rather unexpected. The reason is that so far, we have no mathematical definition of what is meant by the circumference of a circle. (We cannot get the circumference merely by adding the lengths of certain segments, the way we did to get the perimeter of a polygon, because a circle does not contain any segments. Every arc of a circle, no matter how short, is curved at least slightly.) But the remedy is easy; we take the statement

$$p \longrightarrow C$$

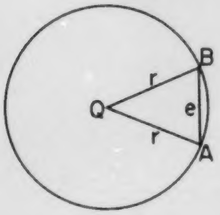
as our definition of C , thus:

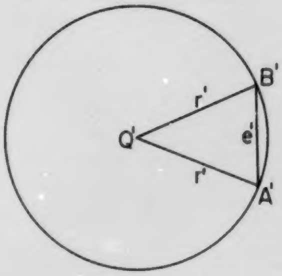
> DEFINITION. The circumference of a circle is the limit of the perimeters of the inscribed regular polygons.

We would now like to go on to define the number π as the quotient of the circumference of a circle divided by its diameter. But to make sure that this definition makes sense, we first need to know that the number $\frac{C}{2r}$ is the same for all circles, regardless of their size. Thus we need to prove the following:

THEOREM 12-17. The quotient of the circumference divided by the diameter, $\frac{C}{2r}$, is the same for all circles.

Proof: We use similar triangles. Given a circle with center Q and radius r , and another circle, with center Q' and radius r' , we inscribe a regular n-gon in each of them. (The same value of n must be used in each circle.)





In the figure we show only one side of each n-gon, with the associated isosceles triangle. Let e and e' be their lengths as shown. Now $\angle AQB \cong \angle A'Q'B'$, because each of these angles has measure $\frac{360}{n}$. Therefore, since the adjacent sides are proportional,

$$\triangle AQB \sim \triangle A'Q'B'$$

by the S.A.S. Similarity Theorem. Therefore $(e,r) \underset{p}{=} (e',r')$ or $(ne,r) \underset{p}{=} (ne',r')$. But ne is the perimeter of the first n-gon, and ne' is the perimeter of the second. We can write, $(p,r) \underset{p}{=} (p',r')$. Now let C and C' be the respective circumferences of the two circles. Then $p \longrightarrow C$ and $p' \longrightarrow C'$, by definition of circumference of a circle. It is plausible that $(C,r) \underset{p}{=} (C',r')$. By alternation we can write this as $(C,C') \underset{p}{=} (r,r')$. It follows that $(C,C') \underset{p}{=} (2r,2r')$. The last proportionality shows the constant of proportionality to be $\frac{C}{2r}$. It is designated by π . We see that the circumference of any circle divided by its diameter is π .

We express the conclusion of Theorem 12-17 in the well-known formula

$$C = 2\pi r .$$

It can be proved that π is not a rational number, that is, it cannot be represented by an expression $\frac{p}{q}$, where p and q are integers. It can, however, be approximated as closely as we please by rational numbers. Some rational approximations to π are

$$3\ ,\quad 3.14\ ,\quad \frac{22}{7}\ ,\quad 3.1416\ ,\quad \frac{355}{113}\ ,\quad 3.1415926535\ .$$

(As a general rule, if there is a choice, you should leave your answers to problems in terms of π .)

Corollary 12-17-1. The circumference of circles are proportional to their radii.

Problem Set 12-6

1. Which is the closer approximation to π , 3.14 or $\frac{22}{7}$?

2. In the following problems C = circumference, r = radius and d = diameter of a circle. Find the indicated parts.

   (a) $r = 7$, $d =$ ____ , $C =$ ____ .
   (b) $C = 36$, $d =$ ____ , $r =$ ____ .
   (c) $d = 15$, $C =$ ____ , $r =$ ____ .
   (d) $r = 6a$, $C =$ ____ , $d =$ ____ .
   (e) $r = x\sqrt{3}$, $C =$ ____ , $d =$ ____ .

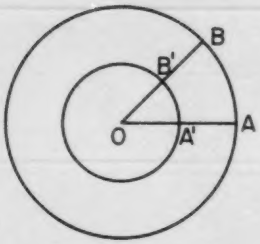
3. C_1 and C_2 are the circumferences of two circles with radii r_1 and r_2 respectively. Fill in the blanks with the appropriate multiplier.

   (a) If $r_1 = 3r_2$, then $C_1 =$ ____ $\cdot\ C_2$.

   (b) If $C_2 = 5C_1$, then $d_2 =$ ____ $\cdot\ d_1$.

   (c) If $d_2 = \frac{1}{2}d_1$, then $C_1 =$ ____ $\cdot\ C_2$.

   (d) If $r_2 = d_1$, then $C_2 =$ ____ $\cdot\ C_1$.

4. 

Two concentric circles have central angles $\angle BOA$ as indicated.

(a) The $m\widehat{BA}$ (in degrees) equals how many times $m\widehat{B'A'}$ (in degrees) ?

(b) If $OA = 2OA'$, then you would expect the length of $\widehat{BA}$ to equal how many times the length of $\widehat{B'A'}$? [Lengths of arcs will be considered more precisely in Section 12-8.]

(c) If $\frac{OA}{OA'} = f$, what would you expect

$$\frac{\text{length of } \widehat{BA}}{\text{length of } \widehat{B'A'}}$$ to equal?

5. Given $T = \{(x,y): x^2 + y^2 = 36\}$

$U = \{(x,y): x^2 + y^2 = 16\}$.

(a) What is the ratio of the circumference of T to the circumference of U ?

(b) What would you expect the ratio of the lengths of the subsets of T and U which are in the union of the positive x-axis, the positive y-axis and Quadrant I to be?

6. The moon is about 240,000 miles from the earth, and its path around the earth is nearly circular. Find the circumference of the circle which the moon describes every month.

7. The earth is about 93,000,000 miles from the sun. The path of the earth around the sun is nearly circular. Find how far we travel every year "in orbit." What is our speed in this orbit in miles per hour?

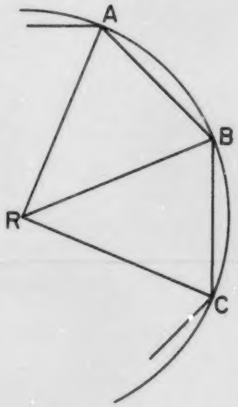
8. A certain tall person takes steps a yard long. He walks around a circular pond close to the edge taking 628 steps. What is the approximate radius of the pond? (Use 3.14 as an approximation for π .)

9. A regular polygon is inscribed in a circle, then another with one more side than the first is inscribed, and so on endlessly, each time increasing the number of sides by one.

   (a) What is the limit of the length of the apothem?
   (b) What is the limit of the length of a side?
   (c) What is the limit of the measure of an angle?
   (d) What is the limit of the perimeter of the polygon?

10. The figure represents part of a regular polygon of which $\overline{AB}$ and $\overline{BC}$ are sides, and R is the center of the circle in which the polygon is inscribed. Copy and complete the table:

Number of sides	m $\angle$ARB or m $\angle$BRC	m $\angle$ABR or m $\angle$CBR	m $\angle$ABC
3	______	______	______
4	______	______	______
5	______	______	______
6	______	______	______
______	45	______	______
9	40	70	140
______	______	______	144
12	______	______	______
15	______	______	______
18	______	______	______
20	______	______	______
24	______	______	______



11. The sides of a regular hexagon are each 2 units long. If it is inscribed in a circle, find the radius of the circle and the apothem of the hexagon.

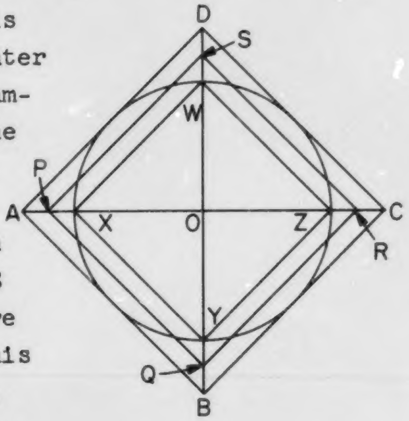
12. The side of a square is 12 inches. What is the circumference of its inscribed circle? Of its circumscribed circle?

13. (a) One circle has a radius of 10 feet. A second circle has a radius one foot longer. How much longer is the circumference of the second circle than that of the first?

    (b) How much longer is the circumference of a circle whose radius is 1001 feet than that of a circle whose radius is one foot shorter?

14. A regular octagon with sides 1 unit long is inscribed in a circle. Find the radius of the circle.

15. In the figure, square XYZW is inscribed in a circle with center O, and square ABCD is circumscribed about this circle. The diagonals of both squares lie in $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$. Given that a square PQRS is formed when the midpoints P, Q, R and S of $\overline{AX}$, $\overline{BY}$, $\overline{CZ}$, and $\overline{DW}$ are joined, is the perimeter of this square equal to, greater than, or less than the circumference of the circle? Let OX = 1 and justify your answer by computation.



12-7. Area of a Circle.

In Chapter 11 we considered areas of polygonal-regions, defined in terms of a basic region, the triangular-region, which is the union of a triangle and its interior. In talking about areas associated with a circle we make a similar basic definition.

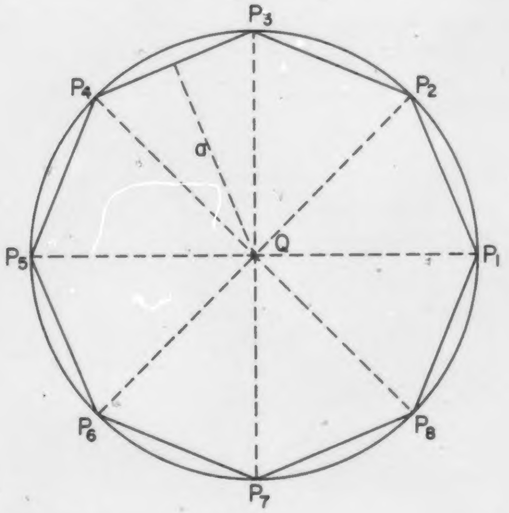
DEFINITION. A circular-region is the union of a circle and its interior.

In speaking of "the area of a triangular-region" we found it convenient to abbreviate this phrase to "the area of a triangle." Similarly, we usually say "the area of a circle" as an abbreviation of "the area of a circular-region."

We now get a formula for the area of a circle. We already have a formula for the area of an inscribed regular n-gon; this is

$$A_n = \frac{1}{2}ap$$

where a is the apothem and p is the perimeter.



In this situation there are three quantities involved, each depending on n, the number of sides. These are p, a and A_n. To get a formula for the area of a circle, we need to find out what limits these quantities have as n becomes large.

(1) What happens to A_n. A_n is always slightly less than the area A of the circle, because there are always some regions that lie inside the circle but outside the regular n-gon.

But the difference between A_n and A is very small when n is very large, because when n is very large the polygonal-region almost fills up the interior of the circle. Thus we expect that

$$A_n \longrightarrow A .$$

But just as in the case of the circumference of the circle, this cannot be proved until we say what we mean by the area of a circle. Here also, the way out is easy:

> *DEFINITION*. The *area of a circle* is the limit of the areas of the inscribed regular polygons.

Thus, $A_n \longrightarrow A$ by definition.

(2) *What happens to* a . The apothem a is always slightly less than r , because either leg of a right triangle is shorter than the hypotenuse. But the difference between a and r is very small when n is very large. Thus, we expect that

$$a \longrightarrow r .$$

(3) *What happens to* p . By definition of C , we have $p \longrightarrow C$.

Fitting together the results in (2) and (3), we get

$$\tfrac{1}{2}ap \longrightarrow \tfrac{1}{2}rC .$$

Therefore

$$A_n \longrightarrow \tfrac{1}{2}rC .$$

But we knew from (1) that

$$A_n \longrightarrow A .$$

Since A_n has only one limit, it follows that

$$A = \tfrac{1}{2}rC .$$

Combining this with the formula $C = 2\pi r$ gives

$$A = \pi r^2 .$$

Thus the formula that you have known for years finally becomes a theorem:

THEOREM 12-18. The area of a circle of radius r is πr^2.

Corollary 12-18-1. The areas of two circles are proportional to the squares of their radii.

Problem Set 12-7

1. Find to the nearest tenth the circumference and area of a circle with radius

   (a) 5 (c) 2.5
   (b) 10 (d) $\sqrt{3}$

2. Find exactly (in terms of π) the circumference and area of a circle with radius

   (a) 1 (c) 7
   (b) 2 (d) $\sqrt{5}$

3. Find the circumference of a circle whose area is

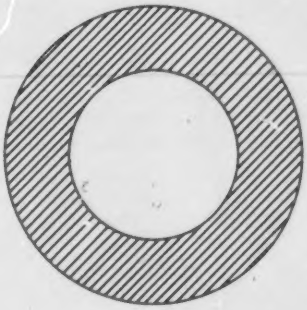
   (a) 25π (c) 25
   (b) 49π (d) 49

4. Find the area of a circle whose circumference is

   (a) 12π (c) 12
   (b) 20π (d) 20

5. (a) Find the area of one face of this iron washer if its diameter is 4 centimeters and the diameter of the hole is 2 centimeters.

   (b) Would the area be changed if the two circles were not concentric?



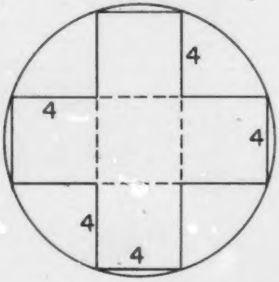
6. The radius of the larger of two circles is three times the radius of the smaller. Find the ratio of the area of the first to that of the second.

7. The circumference of a circle and the perimeter of a square are each equal to 20 inches. Which has the greater area? How much greater is it?

8. Given a square whose side is 10 inches, what is the area between its circumscribed and inscribed circles?

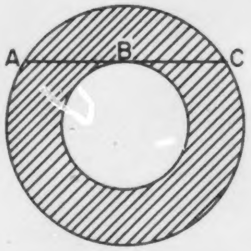
9. An equilateral triangle is inscribed in a circle. If the side of the triangle is 12 inches, what is the radius of the circle? The circumference? The area?

10. The cross inside the circle is divisible into 5 squares. Find the area which is inside the circle and outside the cross.



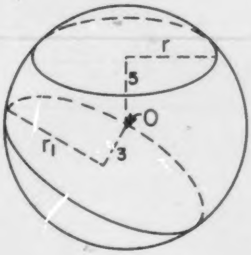
11. Given: Two concentric circles. $\overline{AC}$ is a chord of the larger and is tangent to the smaller at B .

Prove: The area of the ring (annulus) is $\pi(BC)^2$.

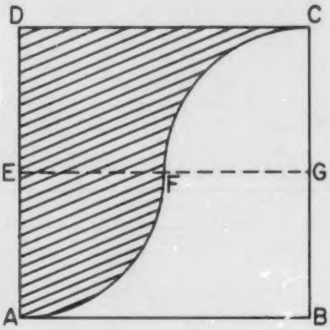


12. In a sphere whose radius is 10 inches, sections are made by planes 3 inches and 5 inches from the center. Which section will be the larger?

Prove that your answer is correct.

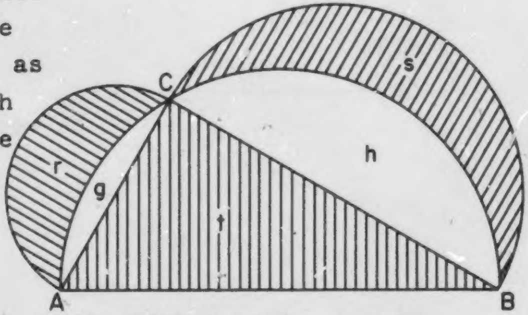


13. In the figure, ABCD is a square in which E, F, G are midpoints of $\overline{AD}$, $\overline{AC}$, and $\overset{\frown}{CB}$, respectively. $\overset{\frown}{AF}$ and $\overset{\frown}{FC}$ are circular arcs with centers E and G respectively. If the side of the square is s, find the area of the shaded portion.



14. In the figure, semicircles are drawn with each side of right triangle ABC as diameter. Areas of each region in the figure are indicated by lower-case letters.

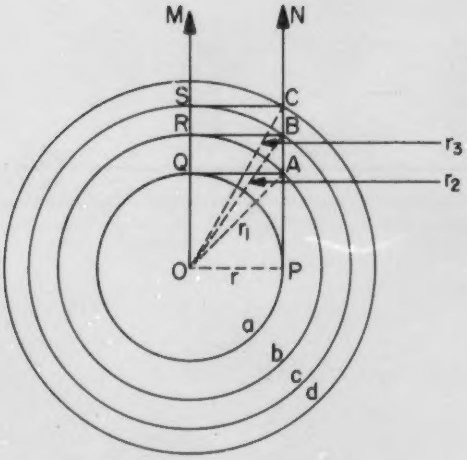
Prove: $r + s = t$.



15. A special archery target, with which a novice can be expected to hit the bulls-eye as often as any ring, is constructed in the following way. Rays $\overrightarrow{OM}$ and $\overrightarrow{PN}$ are parallel. A circle with center O and radius r, equal to the distance between the rays, is drawn intersecting $\overrightarrow{OM}$ at Q . $\overline{QA} \perp \overline{QM}$. Then a circle with center O and radius OA , or r_1 , is drawn intersecting $\overrightarrow{OM}$ in R . This process is repeated by drawing perpendiculars at R and at S , and circles with radii OB and OC . Note that we arbitrarily stop at four concentric circles.

(a) Find r_1 , r_2 , r_3 in terms of r .

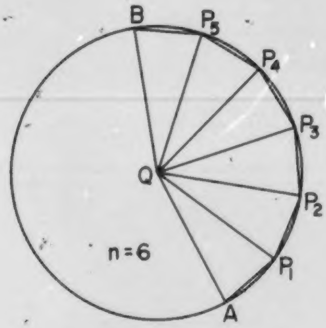
(b) Show that the areas of the inner circle and the three "rings," represented by a, b, c, and d are equal.


M
N
S
C
R
B
r_3
Q
A
r_2
r_1
O
r
P
a
b
c
d

16. An isosceles trapezoid whose bases are 2 inches and 6 inches is circumscribed about a circle. Find the area of the portion of the trapezoid which lies outside the circle.

12-8. Lengths of Arcs. Areas of Sectors.

Just as we define the circumference of a circle as the limit of the perimeters of inscribed regular polygons, so we can define the length of a circular arc as a suitable limit.


B
P_5
P_4
P_3
Q
P_2
n=6
P_1
A

If $\widehat{AB}$ is an arc of a circle with center Q, we take $n - 1$ points $P_1, P_2, \ldots, P_{n-1}$ on $\widehat{AB}$ so that each of the n angles $\angle AQP_1, \angle P_1QP_2, \ldots, \angle P_{n-1}QB$ has measure $\frac{1}{n} \cdot m\widehat{AB}$.

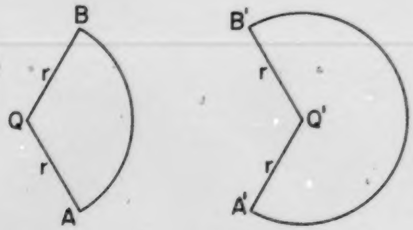
DEFINITION. The length of arc AB is the limit of $AP_1 + P_1P_2 + \ldots + P_{n-1}B$ as we take n larger and larger.

Notation. We sometimes write "$\ell\widehat{AB}$" to mean "the length of $\widehat{AB}$."

It is convenient, in discussing lengths of arcs, to consider an entire circle as an arc whose degree measure is 360. The circumference of a circle can then be considered to be simply the length of an arc whose degree measure is 360.

We now have two types of measure for circular arcs, their degree measure and their length. There is a simple connection between these measures, in the case of congruent circles, namely, that the lengths of arcs on congruent circles are proportional to their degree measures. It is possible to prove this fact, but the proof is very difficult. We prefer to state it as a postulate.

Postulate 31. The lengths of arcs in congruent circles are proportional to their degree measures.



$$(\ell\widehat{AB}, \ell\widehat{A'B'}) \underset{p}{=} (m\widehat{AB}, m\widehat{A'B'}).$$

If we take $\widehat{A'B'}$ to be a semicircle, then $m\widehat{A'B'} = 180$, and $\ell\,\widehat{A'B'} = \pi r$.

Then we may write $(\ell\widehat{AB}, \pi r) \underset{p}{=} (m\widehat{AB}, 180)$.

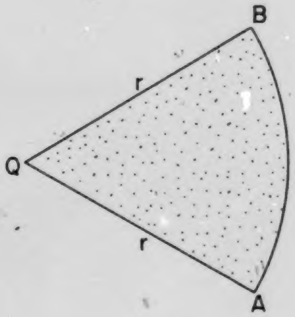
Clearly the constant of proportionality is $\frac{\pi r}{180}$.

THEOREM 12-19. An arc of degree measure q contained in a circle whose radius is r has length L, where

$$L = \frac{\pi r}{180} \cdot q .$$

This result follows from the proportionality, $(L, \pi r) \underset{p}{=} (q, 180)$.

A sector of a circle is a region bounded by two radii and an arc, like this:



More precisely:

DEFINITIONS. If $\widehat{AB}$ is an arc of a circle with center Q and radius r, then the union of all segments $\overline{QP}$, where P is any point of $\widehat{AB}$, is a sector.

$\widehat{AB}$ is the arc of the sector and r is the radius of the sector.

The following theorem is proved just like Theorem 12-18.

THEOREM 12-20. The area of a sector is half the product of its radius and the length of its arc.

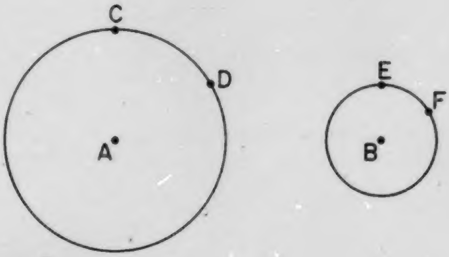
Combined with Theorem 12-19, we get

THEOREM 12-21. The area of a sector of radius r and arc measure q is

$$\frac{\pi r^2}{360} \cdot q .$$

Problem Set 12-8

1. The radius of the circle with center A is 20. The radius of the circle with center B is 10. $m\widehat{CD} = 60$. $m\widehat{EF} = 60$. Is the length of $\widehat{CD}$ greater than, equal to, or less than the length of $\widehat{EF}$?



2. Which has the greater degree measure: an arc of one inch on the equator of the earth or an arc of one inch on a half dollar?

3. Are the degree measures of congruent arcs equal? Are the lengths of congruent arcs equal?

4. Suppose $\widehat{AB}$ on one circle has a larger degree measure than $\widehat{CD}$ on another circle. Does this information permit you to conclude that the length of $\widehat{AB}$ is greater than the length of $\widehat{CD}$? Suppose that you were also told that the length of $\widehat{AB}$ is equal to the length of $\widehat{CD}$. Which circle has the greater radius?

5. The radius of a circle is 15 inches. What is the length of an arc of 60° ? of 90° ? of 72° ? of 36° ?

6. The radius of a circle is 6. What is the area of a sector with an arc of 90° ? of 1° ? of 60° ? of 54° ?

7. What is the area of a sector whose arc has degree measure 90° and arc length 3π ?

8. What is the length of $\overset{\frown}{AB}$ in the circle with center O if m $\angle AOB = 60$ and the area of the sector AOB is 6π ?

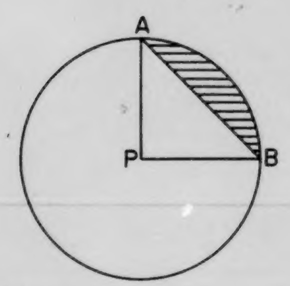
9. If the length of a 60° arc is one centimeter, find the radius of the arc. Also find the length of the chord of the arc.

10. In a circle of radius 2 , a sector has area π . What is the measure of its arc?

11. Given $S = \{(x,y,z): x^2 + y^2 + z^2 = 25\}$
and $P = \{(x,y,z): z = 3\}$.
    (a) Describe the intersection of S and P .
    (b) Compare the circumference of the circle of intersection of S and P and the circumference of a great circle of S .
    (c) Compare the area of the circle of intersection of S and P and the area of a great circle of S .
    (d) Compare the arcs of the circles in Parts (b) and (c) such that all points of the arcs satisfy the conditions that $x \geq 0$ and $y \geq 0$.

12. Find the area of the sector of the circle $x^2 + y^2 = 100$ which has the positive x-axis and the line $y = x$ as partial boundries and which satisfies the condition that $x \geq 0$, $y \geq 0$.

13. A segment of a circle is the region bounded by a chord and an arc of the circle. The area of a segment is found by subtracting the area of the triangle formed by the chord and the radii to its endpoints from the area of the sector. In the figure, m $\angle APB = 90$. If $PB = 6$, then



Area of sector $PAB = \frac{1}{4}\pi \cdot 6^2 = 9\pi$.

Area of triangle $PAB = \frac{1}{2} \cdot 6^2 = 18$.

Area of segment $= 9\pi - 18$ or approximately 10.26

Find the area of the segment if:

(a) m $\angle APB = 60$; $r = 12$.

(b) m $\angle APB = 120$; $r = 6$.

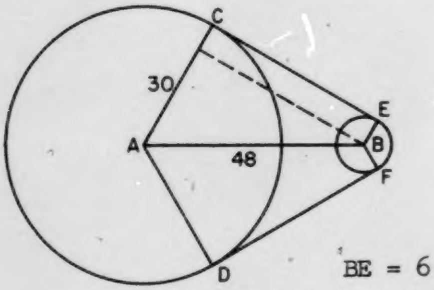
(c) m $\angle APB = 45$; $r = 8$.

14. If a wheel of radius 10 inches rotates through an angle of 36° ,

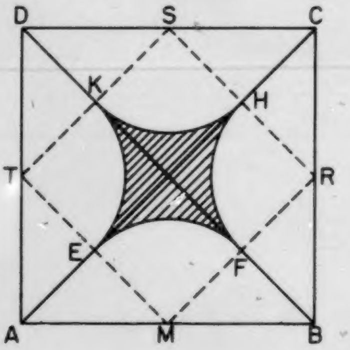
(a) how many inches does a point on the rim of the wheel move?

(b) how many inches does a point on a spoke 5 inches from the center move?

15. A continuous belt runs around two wheels of radii 6 inches and 30 inches. The centers of the wheels are 48 inches apart. Find the length of the belt.



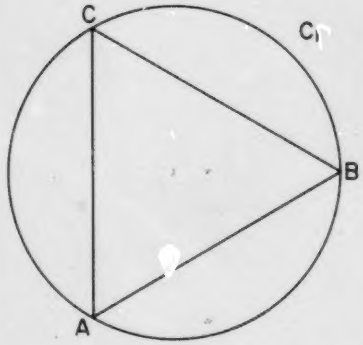
16. In this figure ABCD is a square whose side is 8 inches. With the midpoints of the sides of the square as centers, arcs are drawn tangent to the diagonals. Find the area enclosed by the four arcs.

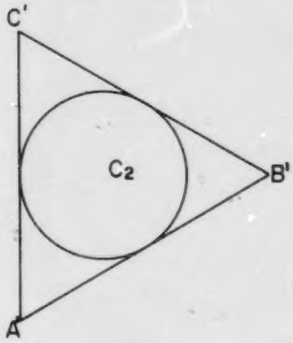


12-9. Inscribed and Circumscribed Circles.

DEFINITIONS. A circle is inscribed in a triangle, or the triangle is circumscribed about the circle, if each side of the triangle is tangent to the circle.

A circle is circumscribed about a triangle, or the triangle is inscribed in the circle, if each vertex of the triangle lies on the circle.

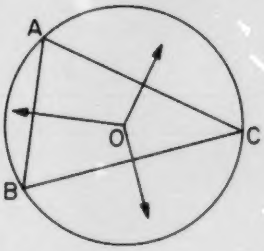




In these figures, $\triangle ABC$ is inscribed in circle C_1 and $\triangle A'B'C'$ is circumscribed about circle C_2. Circle C_2 is inscribed in $\triangle A'B'C'$ and circle C_1 is circumscribed about $\triangle ABC$.

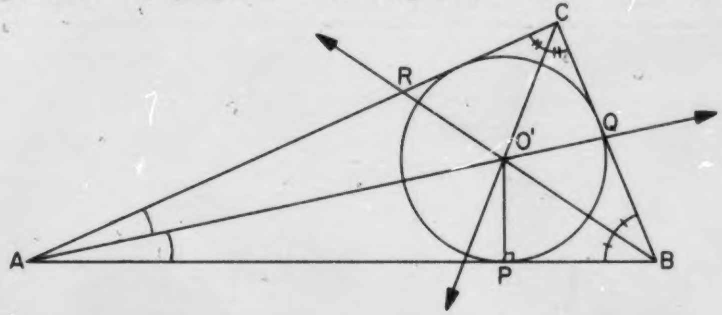
The problem of finding a circle circumscribed about a given triangle can be stated in a slightly different way. If A, B, C are three points, it is natural to inquire if there is a circle which contains all three points, and if so, how many such circles there are. We know that no circle has three collinear points, so we ought to restrict our attention to three points which are noncollinear. Can you see any difference between the problem of circumscribing a circle about a given triangle and that of passing a circle through each of three given noncollinear points?

If we want to circumscribe a circle about a given triangle ABC, we would first have to locate its center O . We start with the requirements that OA = OB = OC . Where do we look for O if OA and OB are to be equal? If OB and OC are to be equal? If OC and OA are to be equal? By Theorem 8-28 we know that the set of all points equally distant from two points is the perpendicular bisector of a line segment joining the given points. We are therefore led to find the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$. Will these perpendiculars meet in one point? In fact, they do, as is proved in Corollary 8-28-1. Since O is the only point that has the property that OA = OB = OC and since OA is the only radius that we can use for our circle we therefore conclude that there is exactly one circle that circumscribes $\triangle ABC$. And thus we have proved



THEOREM 12-22. A triangle has one and only one circumscribed circle. The center of this circle is the intersection of the perpendicular bisectors of the sides of the triangle.

We now turn to circles inscribed in triangles. We look for a circle which has the three sides as tangents. The segment from its center O' to a point of contact is perpendicular to this side. Why? Then the length of this segment is the distance from O' to the sides. Why? Similarly, the lengths of the perpendicular segments from O' to the other sides



would be the distances to those sides. Our problem then is to find a point equally distant from the sides of a triangle. By Theorem 8-29 we know that the set of all points equally distant from the sides of an angle is its midray. Furthermore, by Corollary 8-29-1, we know that the midrays of the angles of a triangle meet in one point. We may therefore find the center of an inscribed circle by finding where the three angle bisectors of the triangle meet. The radius of the circle is the distance from this point to a side. Moreover the circle is unique because its center and radius are unique. We have thus proved

THEOREM 12-23. A triangle has one and only one inscribed circle. The center of this circle is the intersection of the midrays of the angles of the triangle.

Problem Set 12-9

1. Investigate whether the center of the circumscribed circle about a given triangle is in the interior, on, or in the exterior of the triangle. Consider three cases: a triangle whose angles are acute, a triangle with a right angle, and a triangle with an obtuse angle. After making drawings for each case state what seems to be true in each case. Then prove each statement.
2. Must the center of the inscribed circle of a given triangle be in the interior of the triangle? Write an argument to support your answer.
3. The center of the circumscribed circle of a certain triangle is on one side and is 12 inches from each vertex. Find the length of the median to the longest side.
4. Can a circle be circumscribed about a given rectangle? Can a circle be inscribed in a given rectangle? Prove your answers.
5. Can a circle be inscribed in a given rhombus? Prove your answer. Can a circle be circumscribed about a given rhombus?

6. Prove that an equilateral triangle has concentric inscribed and circumscribed circles.

7. Prove that, if a triangle has concentric inscribed and circumscribed circles, then it is equilateral.

8. Prove that a quadrilateral can be circumscribed by a circle if a pair of opposite angles are supplementary. (Hint: Use the indirect method.)

9. Can a circle be circumscribed about a given isosceles trapezoid? Prove your answer. Can a circle be inscribed in a given isosceles trapezoid?

10. The radius of the inscribed circle of a right triangle is 2 inches long. The point of contact on the hypotenuse divides it into two segments whose lengths have the ratio 3:2 . Find the length of the hypotenuse.

11. Consider triangle XYZ with vertices $X(0,0)$, $Y(8,0)$, $Z(5,3)$.

    (a) Find the midpoint and the slope of $\overline{XY}$, and write an equation of the line containing this point and having as its slope the negative reciprocal of the slope of $\overleftrightarrow{XY}$. That is, write an equation of the perpendicular bisector of $\overline{XY}$.

    (b) Similarly, obtain an equation of the perpendicular bisector of $\overline{XZ}$.

    (c) Find the point of intersection, C , of the respective perpendicular bisectors of $\overline{XY}$ and $\overline{XZ}$.

    (d) Obtain an equation of the perpendicular bisector of $\overline{YZ}$. Do the coordinates of C obtained in (c), satisfy this equation? Are the three perpendicular bisectors of triangle XYZ concurrent in the point C ? (If you have not found this to be the case, you should check your work.)

    (e) Show that C is equidistant from the vertices of triangle XYZ . What is the distance CX ?

    (f) Write an equation of the circle having C as center and containing X, Y, Z . This is sometimes called the "circumcircle," and C is the "circumcenter," of the triangle.

12. Consider, again, the triangle XYZ with vertices $X(0,0)$, $Y(8,0)$, $Z(5,3)$.

    (a) Write an equation of the line containing the midpoint of $\overline{XY}$ and the opposite vertex Z. This line contains the median to $\overline{XY}$.

    (b) Write an equation of the line containing the median to $\overline{XZ}$.

    (c) Write an equation of the line containing the median to $\overline{YZ}$.

    (d) Express the three equations of Parts (a), (b), and (c) above in parametric form. For the parameter k take the value $\frac{1}{3}$, or $\frac{2}{3}$, and find the coordinates of the trisection points of each median which is common to all three medians. This point is sometimes called the "centroid" of the triangle.

13. Consider, again, the triangle XYZ with vertices $X(0,0)$, $Y(8,0)$, $Z(5,3)$.

    (a) Write an equation of the line which contains Z and is perpendicular to $\overline{XY}$. This line contains the altitude to $\overline{XY}$.

    (b) Write equations for the lines which contain the altitudes to $\overline{XZ}$ and $\overline{ZY}$, respectively.

    (c) Find the coordinates of the point O which is common to all three of the lines containing the altitudes. This point is sometimes called the "orthocenter" of the triangle. Is the orthocenter of triangle XYZ in the interior or the exterior of the triangle? Sketch a triangle for which the opposite is true.

    (d) It is perhaps surprising to learn that the circumcenter, the centroid, and the orthocenter, of any triangle are collinear. Use the results of Problems 11, 12, and 13 to verify that this is the case for triangle XYZ.

12-10. Summary.

The foot of the perpendicular from the center of a circle to a line is the key to understanding many relations between circles and tangents and between circles and chords. Similarly the foot of the perpendicular from the center of a sphere to a plane helps to explain the relations between a sphere and a tangent plane and the intersection of a sphere and a plane.

We saw some interesting relations between the measures of certain angles related to a circle and the measures of intercepted arcs. These relations are easily remembered by noting that if the vertex of the angle is an interior point of the circle we use one-half of the sum of two arc measures; if on the circle, one-half of an arc measure; if an exterior point, one-half the difference between two arc measures.

We defined the circumference of a circle, the length of an arc, the area of a circle, and the area of a sector as limits of certain measures related to regular polygons. This enabled us to make plausible the formulas used for measuring circumferences, arc lengths, areas of circles and areas of sectors.

Review Problems

(Chapter 12)

1. If C is the circumference of a circle and r is its radius, what is the value of $\frac{C}{r}$?

2. Define (a) the area of a circle.
   (b) the length of an arc of a circle.

3. If the circumference of a circle is 12 inches, the length of its radius will lie between what two consecutive integers?

4. If the diameter of two circles C and C' are d and $2d$ respectively and C makes 10 revolutions in going a distance K, how many revolutions will C' make in going the same distance?

5. What is the radius of a circle if its circumference is equal to its area?

6. If the radius of one circle is 10 times the radius of another, give the ratio of

   (a) their diameters.
   (b) their circumferences.
   (c) their areas.

7. If a regular hexagon is inscribed in a circle of radius 5, what is the length of each side? What is the length of the arc of each side?

8. Show that the area of a circle is given by the formula $A = \frac{1}{4}\pi d^2$, where d is the diameter of the circle.

9. (a) If both a square and a regular octagon are inscribed in the same circle, which has the greater apothem? the greater perimeter?
   (b) Answer the same questions for circumscribed figures.

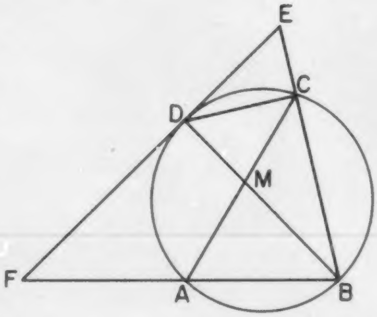
10. From what formula relating to regular polygons is the formula for the area of a circle derived?

11. A wheel has a 20 inch diameter. How many revolutions will it make in going 100 feet?

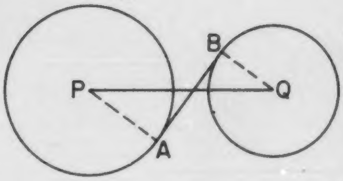
12. The angle of a sector is 10° and its radius is 12 inches. Find the area of the sector and the length of its arc.

13. Prove that the area of an equilateral triangle circumscribed about a circle is four times the area of an equilateral triangle inscribed in the circle.

14. In the figure, $\overleftrightarrow{EF}$ is tangent to the circle at D and $\overrightarrow{CA}$ bisects $\angle BCD$. If $m\overset{\frown}{AB} = 88$ and $m\overset{\frown}{CD} = 62$, find the measure of each arc and each angle indicated in the figure.



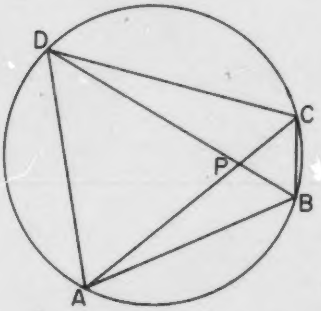
15. The distance between the centers of two circles having radii of 7 and 9 is 20. Find the length of the common internal tangent-segment.



16. Given inscribed quadrilateral ABCD with diagonals intersecting at P.

Prove:

(a) $\triangle APD \sim \triangle BPC$.
(b) $AP \cdot PC = PD \cdot PB$.

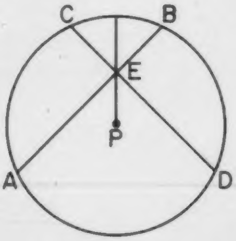


17. Given the circle $C = \{(x,y): x^2 + y^2 = 16\}$.

(a) Are points $A(4,0)$, $B(0,4)$ on the circle? Find the slope of $\overleftrightarrow{AB}$.
(b) Find the midpoint of $\overline{AB}$.
(c) Write an equation of the line containing the midpoint of $\overline{AB}$ and perpendicular to $\overleftrightarrow{AB}$.
(d) Does the line of Part (c) contain the origin? What theorem does this illustrate?
(e) Find the point of intersection of the perpendicular bisector of $\overline{AB}$ with C. This point is the ______ of $\widehat{AB}$.

18. Given: In the figure, P is the center of the circle, and $m\angle AEP = m\angle DEP$.

Prove: $\overline{AB} \cong \overline{CD}$.



REVIEW PROBLEMS

Chapters 10-12

Write (+) if the statement is true and (0) if it is false. Be able to explain why you mark each statement true or false.

1. There can be a region which is completely surrounded by a polygonal-region and which does not contain a point of the polygonal-region.
2. If a polygon is equilateral, it must be a regular polygon.
3. No polygon can be a convex set, but some polygons can enclose a convex set.
4. Each interior angle of a regular pentagon has measure 72 .
5. Every polygonal-region is either a triangular-region or the union of two or more coplanar triangular-regions.
6. The sum of the measures of the face angles of a polyhedral angle can equal 360 .
7. The number of the diagonals from a given vertex of a convex polygon is equal to the number of sides of the polygon.
8. An exterior angle of a regular polygon is congruent to the central angle of the polygon.
9. The sum of the measures of the interior angles of a convex polygon of n sides is $(n-2)180$.
10. The sum of the measures of the exterior angles of a convex polygon, considering one at each vertex, is equal to the sum of the measures of four right angles.
11. If a line intersects a circle in one point, it intersects the circle in two points.
12. It is possible for two triangles to be congruent and to be the boundaries of triangular regions with different areas.
13. The area of a right triangular region is one-half the product of the length of the hypotenuse and the length of the shorter leg.

14. The area of the interior of a parallelogram is the product of the lengths of any two consecutive sides.

15. If $A(0,0)$ and $B(0,6)$ are the endpoints of a diameter of a circle, then $C(3,3)$ is a point on the circle.

16. If two parallelograms have congruent altitudes, the areas of their interiors are proportional to the lengths of their bases.

17. If a plane intersects a sphere in at least two points, the intersection is a line.

18. If a sphere and a circle have the same center and if they intersect, then the intersection is a circle.

19. If a line is tangent to a circle, it is perpendicular to the plane of the circle.

20. A line which is perpendicular to and bisects a chord of a circle contains the center of the circle.

21. The set $\{(x,y,z): x^2 + y^2 + z^2 = 9\}$ is a sphere with center at the origin and with a radius equal to 9.

22. If $C = \{(x,y): x^2 + y^2 = 25\}$, then the line $\{(x,y): y = 5\}$ is tangent to C.

23. The distance between the point $A(1,2,3)$ and the point $B(0,-4,-1)$ is $\sqrt{53}$.

24. Any quadrilateral can be inscribed in some circle.

25. If the lateral edge of a prism is congruent to the altitude of the prism, the prism is a right prism.

26. Any two great circles of a sphere intersect.

27. Two tangent circles are externally tangent only if their centers lie on opposite sides of each common tangent line.

28. The point of tangency of two tangent circles is collinear with the centers of the two circles.

29. The areas of two similar polygons are proportional to the squares of the lengths of any two corresponding sides.

30. The apothem of a regular hexagon of side s is equal to the altitude of an equilateral triangle of side s.

31. The radius of a circle is congruent to the median on the hypotenuse of a right triangle inscribed in the circle.

32. In a circle of radius 12, an inscribed angle of 135° intercepts an arc of measure 9π.

33. A line can intersect a sphere in exactly one point.

34. Two planes tangent to the same sphere must intersect.

35. If two chords intersect within a circle, the difference in the lengths of the segments of one chord is equal to the difference in the lengths of the segments of the other.

36. If the lateral edge of a parallelepiped whose base is a square is congruent to a side of the square base, then the parallelepiped is a cube.

37. If a line intersects the exterior of a sphere, then it must intersect the sphere.

38. Concentric circles have concurrent diameters.

39. A sphere has radius 5. If a plane 3 units from the center intersects the sphere in a circle, the radius of this circle is 4.

40. The plane $\{(x,y,z): z = 7\}$ intersects the sphere $\{(x,y,z): x^2 + y^2 + z^2 = 16\}$ in a circle.

41. Point $(2,3)$ lies on the circle $\{(x,y): x^2 + y^2 = 5\}$.

42. The product of a secant-segment and its external secant-segment is constant for any given circle and exterior point.

43. A pyramid is a regular pyramid if the foot of the perpendicular from its vertex is one of the vertices of the base.

44. The formula $A = \frac{1}{2}ap$ is the formula for the lateral area of any pyramid.

45. $\pi = 3.14159$.

46. The degree measure of a minor arc is the same as the measure of the central angle which intercepts it.

47. All arcs with the same degree measure have the same length if they are all of the same circle.

48. Every circle forms with its interior a polygonal-region.

49. The measure of a tangent-chord angle is one-half the measure of its intercepted arc.

50. An arc whose degree measure is 120 has length $\frac{2}{3}\pi$ if the length of its radius is one.

51. Line $y = 2x$ intersects the circle $x^2 + y^2 = 100$ in the points with coordinates $(2\sqrt{5}, 4\sqrt{5})$ and $(-2\sqrt{5}, -4\sqrt{5})$.

52. $(x - 2)^2 + (y + 5)^2 = 25$ is a circle with center at $(2,5)$.

53. If the measure of the radius of a circle is one, its circumference is 2π and its area is π.

54. The median of an isosceles triangle is parallel to the base.

55. If two chords of a circle bisect each other, then each must be a diameter of the circle.

56. The quadrilateral with vertices $(0,0)$, $(4,0)$, $(4,4)$, $(0,4)$ is a square.

57. In a sphere the planes of great circles are parallel.

58. A unit-square must have a measure of one inch for each side.

59. Angles inscribed in the same arc are congruent.

60. The area of a circle is smaller than the area of any regular polygon in which the circle is inscribed.

61. The area of a trapezoidal-region is the product of the lengths of its altitude and its median.

62. A rectangular prism whose bases are squares, and each of whose lateral edges is twice as long as the side of the base, has a total surface area which is ten times the area of a base.

63. If two triangles have bases of the same length and altitudes of the same length, they have the same area and are similar polygons.

64. If a circle is inscribed in a parallelogram, then the parallelogram must be equilateral.

65. If the degree measure of one arc of a circle is twice the degree measure of a second arc, then the chord associated with the first arc is twice as long as the chord associated with the second arc.

66. If a set of points is a quadrilateral, then the points are coplanar.

67. It is possible for the incenter, the circumcenter, and the orthocenter of a triangle to be the center of the same circle.

68. The square of the length of a tangent-segment from a given exterior point is equal to the product of the lengths of any secant-segment from that point and the length of its external segment.

69. The points of a circle are said to be collinear.

70. If the circumference of a circle is 12π, then the area is 144π.

71. The area of a sector of radius r whose arc has a degree measure q is $\frac{q}{360} \cdot \pi r^2$.

72. The point which is equally distant from all three sides of a triangle, is the intersection of the midrays of the angles of the triangle.

73. The circumference of a circle is the limit of the perimeters of the circumscribed regular polygons.

74. If two parallelograms have the lengths of the base and altitude of one proportional to the lengths of the base and altitude of the other, the parallelograms are similar.

75. A quadrilateral whose vertices have xy-coordinates (0,0), (3,0), (4,2), (1,2) is a parallelogram.

76. All bisectors of the interior angles of a regular polygon intersect in a single point.

77. The perpendicular bisectors of the sides of a triangle are concurrent.

78. A triangle has one and only one circumscribed circle.

79. The center of a circle circumscribed about a right triangle is not in the interior of the triangle.

80. The least number of faces a polyhedron can have is three.

81. If the perimeter of a regular hexagon is p, then the radius of the inscribed circle is $\frac{\sqrt{3}\,p}{12}$.

82. There are only five types of regular polyhedrons.

83. The point $(4,-3)$ is a point of a circle whose center is the point $(0,0)$ and whose radius is 4.

84. The radii of two circles are 3 and 5; the radius of a circle whose area is equal to the sum of the areas of these circles is $\sqrt{34}$.

85. The face angles of a trihedral angle may have measures 120, 2.5, and 90.

86. A regular polyhedron having as many faces as there are months in the year is called a dodecahedron.

87. If $AB = CD$, then $(\overrightarrow{A,B}) \doteq (\overrightarrow{C,D})$.

88. The addition of directed line segments is commutative.

89. Vector addition is commutative.

90. The vector $[6,3]$ determines a unique directed line segment.

91. The vector $[6,3]$ is the additive inverse of $[-6,-3]$.

92. If $\vec{a} \neq \vec{b}$, then $|\vec{a}|$ cannot equal $|\vec{b}|$.

93. If $\vec{a} + \vec{b} = \vec{b} + \vec{a}$, we can conclude $\vec{a} - \vec{b} = \vec{b} - \vec{a}$.

94. The scalar product of two vectors is commutative.

95. The scalar product of two vectors is not zero if the vectors are perpendicular.

96. The origin of $(\overrightarrow{A,B})$ is A .

97. The terminus of $(\overrightarrow{A,B})$ is B .

98. If $(\overrightarrow{A,B}) + (\overrightarrow{C,D}) = (\overrightarrow{A,X})$ and $(\overrightarrow{C,D}) + (\overrightarrow{A,B}) = (\overrightarrow{C,Y})$ then $(\overrightarrow{A,X}) \doteq (\overrightarrow{C,Y})$.

99. If $(\overrightarrow{A,B}) = (\overrightarrow{C,D})$ then $A = C$.

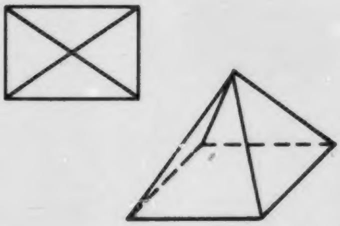
100. If $(\overrightarrow{A,B})$ is not equivalent to $(\overrightarrow{C,D})$, then the vectors they represent cannot be equal.

Appendix V

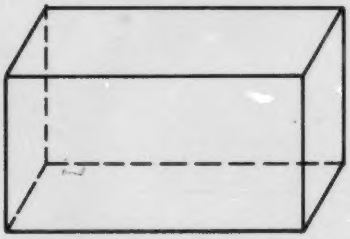
HOW TO DRAW PICTURES OF SPACE FIGURES

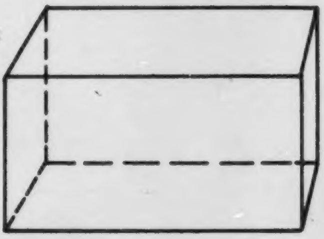
Simple Drawing.

A course in mechanical drawing is concerned with precise representation of physical objects seen from different positions in space. In geometry we are concerned with drawing only to the extent that we use sketches to help us do mathematical thinking. There is no one correct way to draw pictures in geometry, but there are some techniques helpful enough to be in rather general use. Here, for example, is a technically correct drawing of an ordinary pyramid, for a person can argue that he is looking at the pyramid from directly above. But careful ruler drawing is not as helpful as this very crude free-hand sketch. The first drawing does not suggest 3-space; the second one does.



The first part of this discussion offers suggestions for simple ways to draw 3-space figures. The second part introduces the more elaborate technique of drawing from perspective. The difference between the two approaches is suggested by these two drawings of a rectangular box.

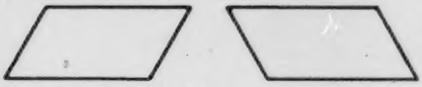




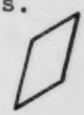
In the first drawing the base is shown by an easy-to-draw parallelogram. In the second drawing, the front base edge and the back base edge are parallel, but the back base edge is drawn shorter under the belief that the shorter length will suggest "more remote".

No matter how a rectangular box is drawn, some sacrifices must be made. All angles of a rectangular solid are right angles, but in each of the drawings shown on the previous page two-thirds of the angles do not come close to indicating ninety degrees when measured with a protractor. We are willing to give up the drawing of right angles that look like right angles in order that we make the figure as a whole more suggestive.

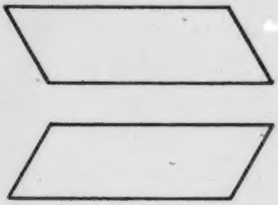
You already know that a plane is generally pictured by a parallelogram.

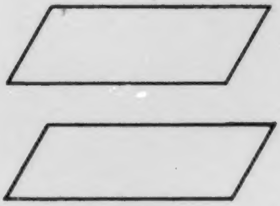


It seems reasonable to draw a horizontal plane in either of the ways shown, and to draw a vertical plane like this.

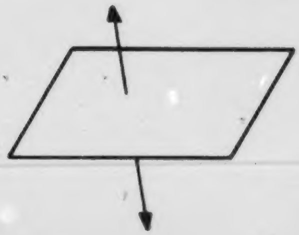


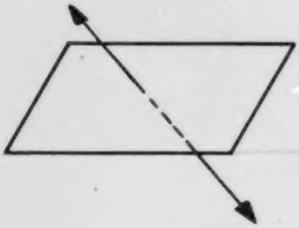
If we want to indicate two parallel planes, however, we can not be effective if we just draw any two "horizontal" planes. Notice how the drawing to the right below improves upon the one to the left. Perhaps you prefer still another kind of drawing.



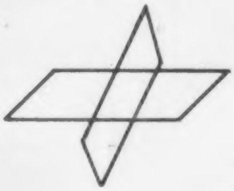


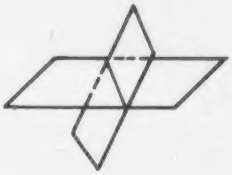
Various devices are used to indicate that one part of a figure passes behind another part. Sometimes a hidden part is simply omitted, sometimes it is indicated by dotted lines. Thus, a line piercing a plane may be drawn in either of the two ways:

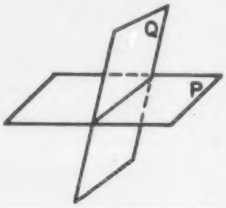


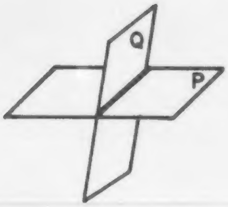


Two intersecting planes are illustrated by each of these drawings.

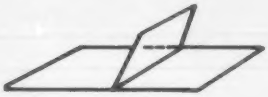






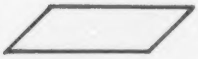


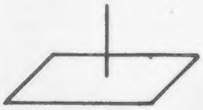
The second is better than the first because the line of intersection is shown and parts concealed from view are dotted. The third and fourth drawings are better yet because the line of intersection is visually tied in with plane P as well as plane Q by the use of parallel lines in the drawing.
Here is a drawing which has the advantage of simplicity and the disadvantage of suggesting one plane and one halfplane.

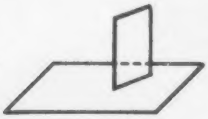


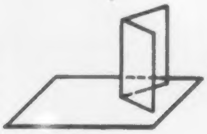
In any case a line of intersection is a particularly important part of a figure.

Suppose that we wish to draw two intersecting planes each perpendicular to a third plane. An effective procedure is shown by this step-by-step development.

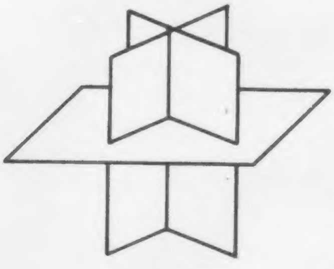


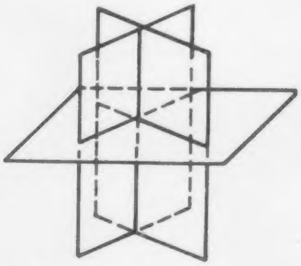




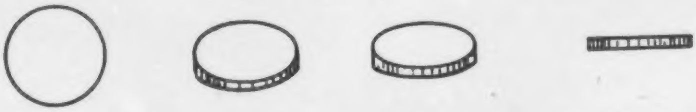


Notice how the last two planes drawn are built on the line of intersection. A complete drawing showing all the hidden lines is just too involved to handle pleasantly. The picture below is much more suggestive.

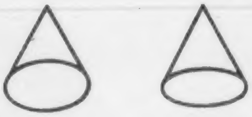




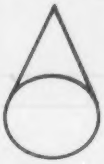
A dime, from different angles, looks like this:



Neither the first nor the last is a good picture of a circle in 3-space. Either of the others is satisfactory. The thinner oval is perhaps better to use to represent the base of a cone.

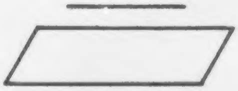


Certainly nobody should expect us to interpret the figure shown below as a cone.

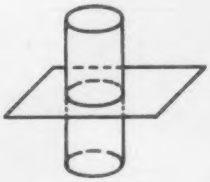


A few additional drawings, with verbal descriptions, are shown.

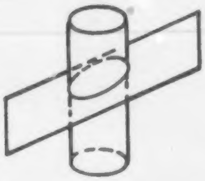
A line parallel to a plane.



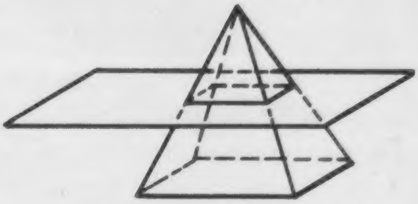
A cylinder cut by a plane parallel to the base.



A cylinder cut by a plane not parallel to the base.



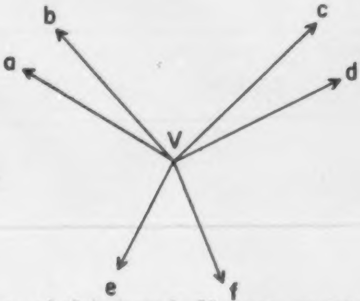
A pyramid cut by a plane
parallel to the base.

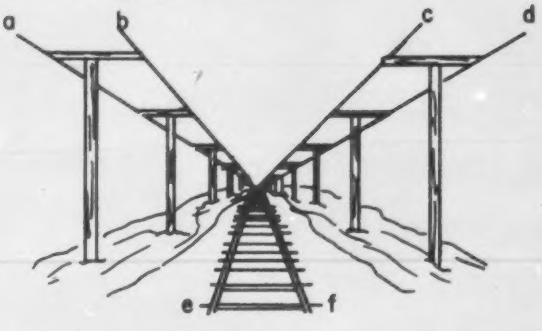


It is important to remember that a drawing is not an end in itself but simply an aid to our understanding of the geometrical situation. We should choose the kind of picture that will serve us best for this purpose, and one person's choice may be different from another.

Perspective.

The rays a , b , c , d , e , f in the left-hand figure below suggest coplanar lines intersecting at V ; the corresponding rays in the right-hand figure suggest parallel lines in a three-dimensional drawing. Think of a railroad track and telephone poles as you look at the right-hand figure.





The right-hand figure suggests certain principles which are useful in making perspective drawings.

(1) A set of parallel lines which recede from the viewer are drawn as concurrent rays; for example, rays a , b , c , d , e , f. The point on the drawing where the rays meet is known as the "vanishing point".

(2) Congruent segments are drawn smaller when they are further from the viewer. (Find examples in the drawing.)

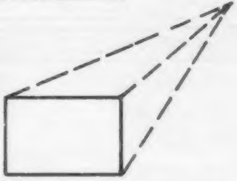
(3) Parallel lines which are perpendicular to the line of sight of the viewer are shown as parallel lines in the drawing. (Find examples in the drawing.)

A person does not need much artistic ability to make use of these three principles.

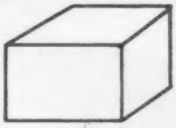
The steps to follow in sketching a rectangular solid are shown below.

Draw the front face as a rectangle.

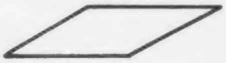
Select a vanishing point and draw segments from it to the vertices. Omit segments that cannot be seen.



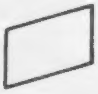
Draw edges parallel to those of the front face. Finally erase lines of perspective.



Under this technique a single horizontal plane can be drawn as the top face of the solid shown above.



A single vertical plane can be represented by the front face or the right-hand face of the solid.



After this brief account of two approaches to the drawing of figures in 3-space we should once again recognize the fact that there is no one correct way to picture geometric ideas. However, the more "real" we want our picture to appear, the more attention we should pay to perspective. Such an artist as Leonardo da Vinci paid great attention to perspective. Most of us find this done for us when we use ordinary cameras.

See some books on drawing or look up "perspective" in an encyclopedia if you are interested in a detailed treatment.

Appendix VII

SURFACE AREA AND VOLUME

Introduction.

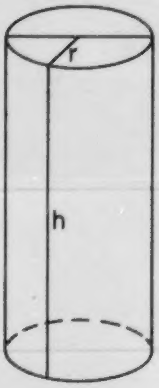
In your study of informal geometry you learned formulas for finding surface areas and volumes of familiar figures. You may recall that a sphere of radius r has volume $\frac{4}{3}\pi r^3$, for instance, and that its surface area is given by $4\pi r^2$. This course in geometry that you have just completed covered formally most of the other topics that you met in informal geometry, and you may wonder why topics of surface area and volume were omitted. The reason has to do with a branch of higher mathematics known as "integral calculus." Until the integral calculus was invented, in the seventeenth century, the study of area and volume was sketchy. There were no satisfactory definitions of area and volume and no systematic ways of finding them. The subject consisted of the discovery and study of formulas for the areas and volumes of individual figures. Moreover the derivation of these formulas, in pre-calculus geometry, are almost without exception either very long and difficult to follow, or logically unsound. It seemed unfair to inflict this kind of study on the high school student when (a) there are parts of geometry from which he could profit more and (b) he will see a suitable development of this subject when (and if) he studies calculus.

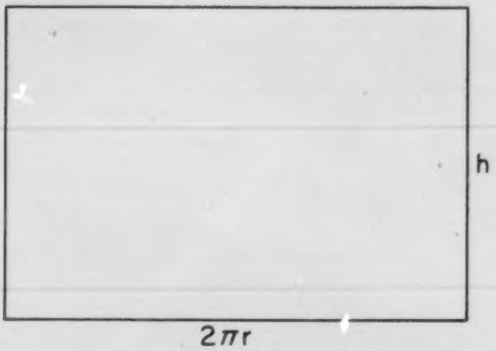
Surface Area.

We shall discuss surface area in an informal way, more as though we were talking about physical objects than mathematical ones.

It is easy to find the surface area of solids such as prisms and pyramids, because their surfaces are composed of plane figures, namely polygons. There are a few other types of solids, whose surfaces are not made up of plane figures, whose surface area can be found by finding an equivalent plane figure.

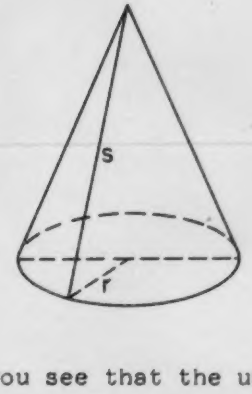
The cylinder and the cone are such figures. If you imagine slitting a cylinder and unrolling its surface onto a plane, what figure do you think is obtained?

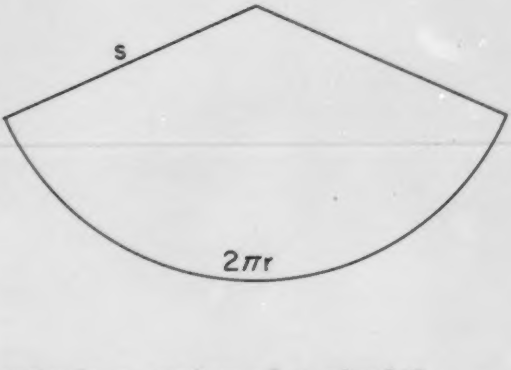




Can you see that it is a rectangle whose altitude is the altitude of the cylinder and whose base is the circumference of the base of the cylinder? If r is the radius of the base of a cylinder and h is its height, then the surface area of the cylinder (called its lateral surface) is equal to the area of the rectangle obtained by unrolling the cylinder. Thus, its lateral surface is $2\pi rh$.

Cones can be treated in a similar way. If a cone is slit properly, it can be unrolled to lie flat.



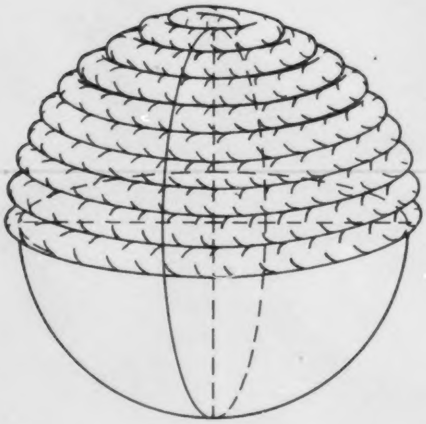


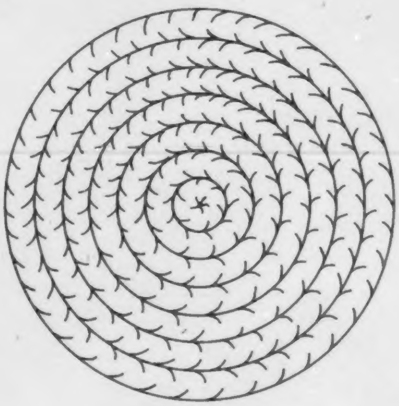
Can you see that the unrolled cone is a sector of a circle? This procedure reduces the problem of finding the surface area of a cone to that of finding the area of a sector.

This latter problem has already been solved (Theorem 12-20). All we need to know in any given case is the radius of the circle and the length of the intercepted arc. In the case of a sector obtained by unrolling a cone, the radius of the sector is the slant height of the cone and the length of the intercepted arc is the circumference of the base of the cone. Therefore, the formula for the lateral surface of a cone is $\frac{1}{2}\ell C$, where ℓ is its slant height and C is the circumference of its base. Another important surface for which there is an area formula is the sphere. The formula is $4\pi r^2$, where r is the radius of the sphere.

It is natural to try to derive this formula by slitting the sphere and unrolling it onto some plane figure. However, it has been proved in higher mathematics that it is impossible to flatten out the sphere in this way. (Can you see any connection between this statement and the fact that maps of large portions of the earth have to distort the shapes of the regions which they depict?)

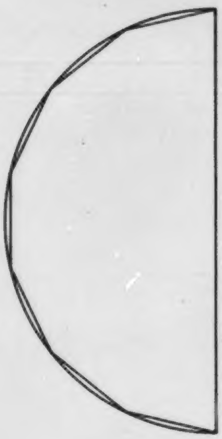
There is a simple experiment which you can perform to test the formula $4\pi r^2$ for the surface area of a sphere. Wind a string around a sphere, in a spiral, until an entire hemisphere is covered, and measure the length of the string required. Then

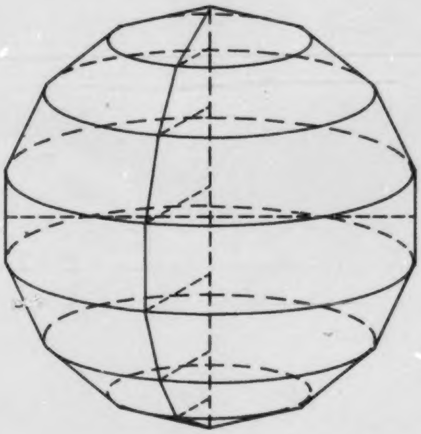




wind a string (of the same diameter) in a planar spiral until it just covers the inside of a circle of the same radius as the given sphere, and measure the length of the string required. If your work is accurate the length of string required to cover the hemisphere will be twice the length required to cover the circle. Since the area of the circle is πr^2, a hemisphere should have surface area $2\pi r^2$, and the whole sphere should have surface area $4\pi r^2$.

A more sophisticated approach to deriving the formula for the surface area of a sphere is to approximate the surface of a sphere by revolving suitable chords around a diameter, as shown below.





This makes it possible to find the surface of a sphere, approximately, in terms of surfaces of portions of cones. A limiting process then yields the formula $4\pi r^2$ for the surface area of a sphere.

Possible Definitions of Volume.

It is a strange fact that mathematicians discovered formulas for the volume of many figures long before they knew what volume was, or at least before they had a formal definition of volume. One way of understanding this is to observe that they had some general notions as to what should be true about volumes, which, in the case of some of the simpler figures, were sufficient to lead to definite formulas.

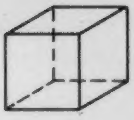
Let us draw up a list of some requirements that are reasonable to impose on any possible definition of volume.

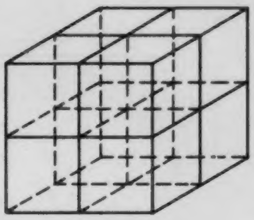
1. It is reasonable to expect that the volume of a solid should be a non-negative real number.

2. It is reasonable to expect that if a solid is partitioned into several parts and that if each of the solids involved has a volume, then the volume of the original solid should equal the sum of the volumes of the parts.

3. It is reasonable to expect that congruent solids should have equal volumes.

4. It is an important fact about volume (which may or may not seem reasonable) that if solid S is similar to solid S', and if the proportionality factor is k, then the volume of solid S' is k^3 times the volume of solid S. For instance, consider two cubes, S and S', such that the edge of S' is twice as long as the edge of S. Then S is similar to S' and the proportionality factor is 2.



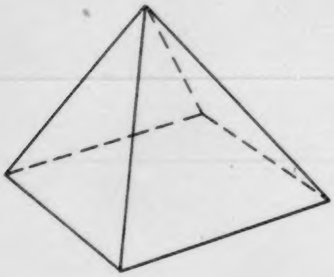


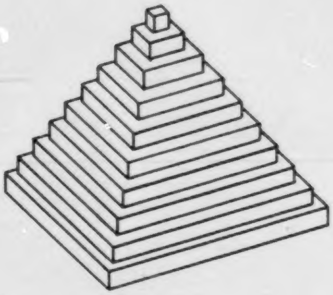
   Notice that S' can be partitioned into eight cubes each congruent to S. Is it not reasonable to expect that the volume of S' should be 2^3 times the volume of S?

5. It is reasonable to require that the volume of a rectangular parallelepiped should be the product of its altitude by the area of its base.

Cavalieri's Principle.

Even though we have not been able to define "volume", there are some situations in which we can reasonably say that two solids have the same volume. We are going now to illustrate one important case of this sort. It will help us understand the case in question if we first think of a physical model. We can make an approximate model of a square pyramid by forming a stack of thin cards, cut to the proper size, like this:



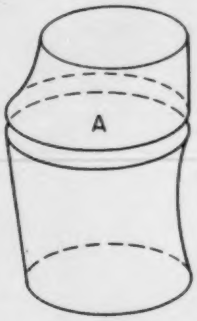


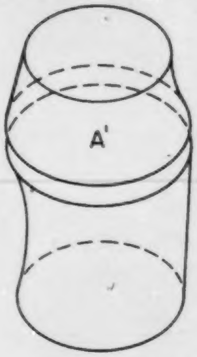
The figure on the left represents the exact pyramid, and the figure on the right is the approximate model made from cards.

Now, suppose we drill a narrow hole in the model, from the top to some point of the base, and insert a thin rod so that it goes through every card in the model. This enables us to tilt the rod in any way we want, keeping its bottom end fixed on the base. Such tilting changes the shape of the model, but not its volume. The reason is that its volume is simply the total volume of the cards; and this total volume does not change as the cards slide along each other.

The same principle applies more generally. Suppose we have two solids with bases in a plane which we shall think of as horizontal. If all horizontal cross-sections of the two solids at the same level have the same area, then the two solids have the same volume. To see this, observe that if we make a card model of each of the solids, then each card in the first model has exactly the same volume as the corresponding card in the second model.

Therefore, the volumes of the two models ought to be the same. The approximation given by the models is as close as we please, if only the cards are thin enough. Therefore, the volumes of the two solids that we started with ought to be the same.





The principle involved here is called Cavalieri's Principle. We have not proved it; we have merely been explaining why it is reasonable. Let us state it explicitly.

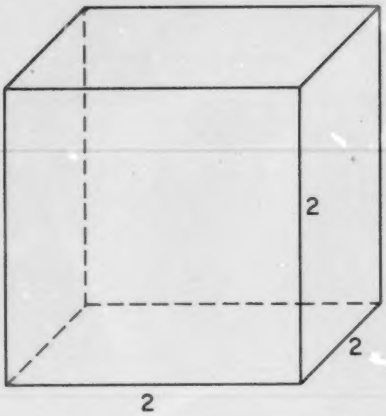
Cavalieri's Principle: Given two solids and a plane. If, for every plane which intersects the solids and is parallel to the given plane, the two intersections have equal areas, then the two solids have equal volumes.

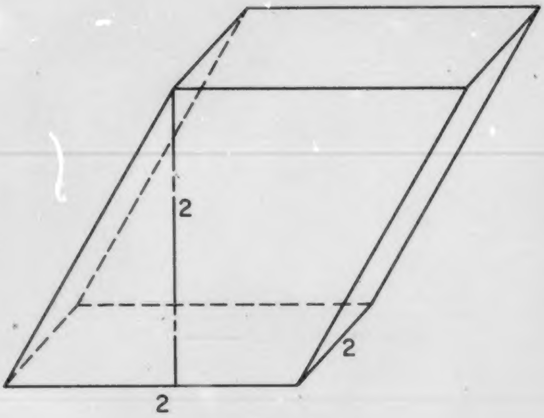
Cavalieri's Principle can be used as a key to the calculation of volumes, as we shall see in the next section.

Prisms and Pyramids.

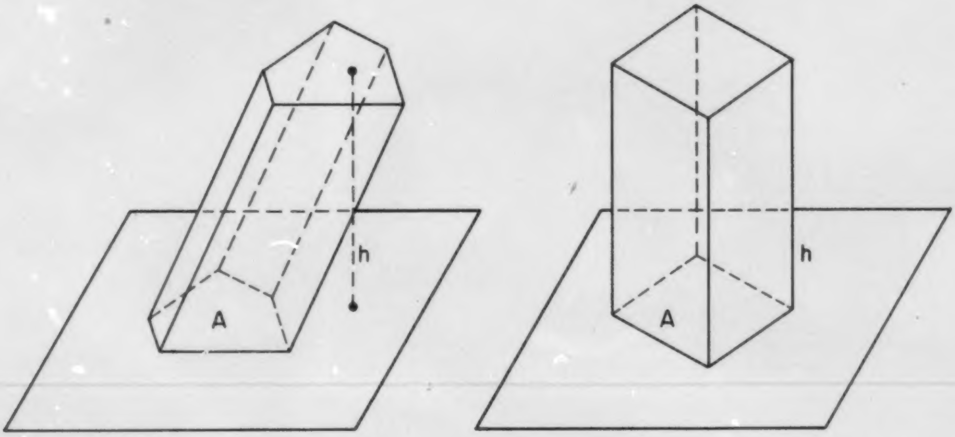
The formula for the volume of a rectangular parallelepiped also applies to general parallelepipeds.

This can be seen by using Cavalieri's Principle to compare the volume of a parallelepiped with that of the appropriate rectangular parallelepiped.





The volume of any prism is the product of its altitude and the area of its base.

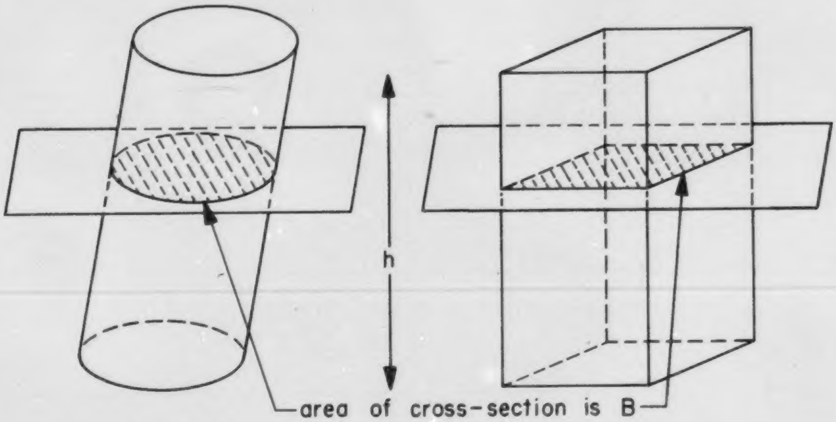


The volume of a pyramid is given by $\frac{1}{3}$ hB where h is its altitude and B is the area of its base. Notice the occurrence of the factor $\frac{1}{3}$ in this formula. Perhaps it reminds you of the factor $\frac{1}{2}$ which occurs in the formula $\frac{1}{2}$ hb for the area of a triangle. These factors are indeed analogous, and we now try to show how. In the derivation of the formula for the area of a triangle of altitude h and base b an auxiliary parallelogram was introduced, also having altitude h and base b, and which could be dissected into two triangles, each congruent to the original one. Since the parallelogram was known to have area hb, the formula $\frac{1}{2}$ hb was readily deduced. In the case of a triangular pyramid with altitude h and base area B, the auxiliary figure is a triangular prism with the same base and altitude, and which can be dissected into three triangular pyramids, each having the same volume as the given one. Since the prism has volume hB, the formula $\frac{1}{3}$ hB is readily deduced.

The formula for the volume of any pyramid, not necessarily triangular, is also $\frac{1}{3}$ hB. It can be derived by dissecting the given pyramid into triangular pyramids and observing that the volume of the original figure is the sum of the volumes of the auxiliary pyramids.

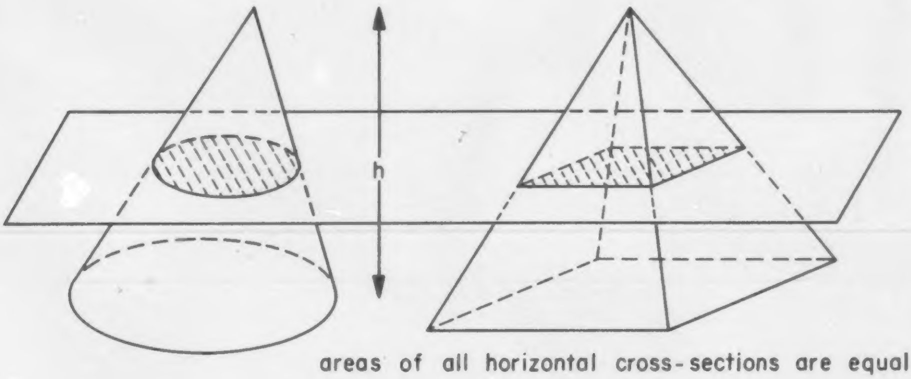
Cylinders. Cones and Spheres.

One way of finding the volume of a cylinder is to introduce a suitable prism and use Cavalieri's Principle.



By referring to a prism having the same altitude h and cross-sectional area B as our cylinder, we see that the volume of the cylinder is hB.

One way of finding the volume of a cone is to introduce a suitable pyramid and use Cavalieri's Principle.



areas of all horizontal cross-sections are equal

By referring to a pyramid having the same altitude h as our cone, and equal cross-sectional area at corresponding levels, we can infer that the volume of the cone is $\frac{1}{3}hB$, where B is the area of its base.

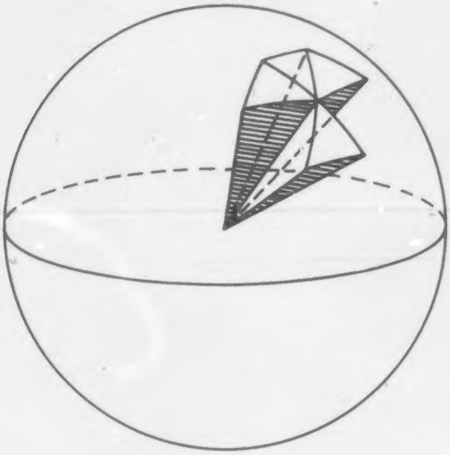
Another important solid for which there is a volume formula is the interior of a sphere. The volume of such a region is $\frac{4}{3}\pi r^3$, where r is the radius of the sphere. Let us see if we can find some justification for this statement.

If planes are drawn through the center of the sphere, the sphere is partitioned into solids which are very much like pyramids. These solids have curved bases, so the formula we have for the volume of a pyramid ought not be used for finding their volumes. However, if enough planes are drawn, the base of any one of these solids is almost flat, its altitude is almost equal to the radius of the sphere, so it is not unreasonable to believe that its volume is given by

$$\frac{1}{3}r \cdot B$$

where B is the surface area of its base. Therefore, the total volume of the sphere appears to be the sum of all these volumes. A little algebraic manipulation shows that their sum is $\frac{1}{3}r$ times the sum of all the areas B. Thus, the volume of the sphere appears to be $\frac{1}{3}rS$, where S is the surface area of the sphere.

Since S is $4\pi r^2$, it therefore appears that $\frac{4}{3}\pi r^3$ is the correct formula.



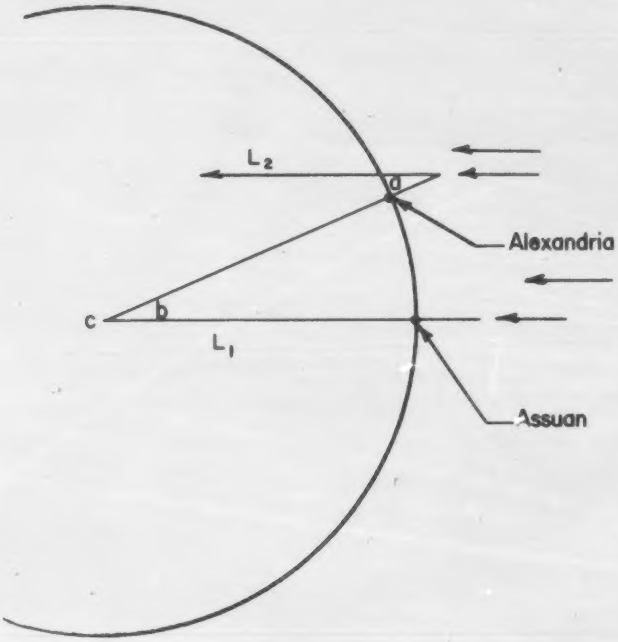
440

Appendix VIII

HOW ERATOSTHENES MEASURED THE EARTH

The circumference of the earth, at the equator, is about 40,000 kilometers, or about 24,900 miles. Christopher Columbus appears to have thought that the earth was much smaller than this. At any rate, the West Indies got their name, because when Columbus reached them, he thought that he was already in India. His margin of error, therefore, was somewhat greater than the width of the Pacific Ocean.

In the third century B.C., however, the circumference of the earth was measured, by a Greek mathematician, with an error of only one or two per cent. The man was Eratosthenes, and his method was as follows:



It was observed that at Assuan on the Nile, at noon on the Summer Solstice, the sun was exactly overhead. That is, at noon of this particular day, a vertical pole cast no shadow at all, and the bottom of a deep well was completely lit up.

In the figure, C is the center of the earth. At noon on the Summer Solstice, in Alexandria, Eratosthenes measured the angle marked *a* on the figure, that is, the angle between a vertical pole and the line of its shadow. He found that this angle was about $7^\circ 12'$, or about $\frac{1}{50}$ of a complete circumference.

Now, the sun's rays, observed on earth, are very close to being parallel. Assuming that they are actually parallel, it follows when the lines L_1 and L_2 in the figure are cut by a transversal, alternate interior angles are congruent. Therefore, $\angle a \cong \angle b$. Therefore, the distance from Assuan to Alexandria must be about $\frac{1}{50}$ of the circumference of the earth.

The distance from Assuan to Alexandria was known to be about 5,000 Greek stadia. (A *stadium* was an ancient unit of distance.) Eratosthenes concluded that the circumference of the earth must be about 250,000 stadia. Converting to miles, according to what ancient sources tell us about what Eratosthenes meant by a stadium, we get 24,662 miles.

Thus, Eratosthenes' error was well under two per cent. Later, he changed his estimate to an even closer one, 252,000 stadia, but nobody seems to know on what basis he made the change. On the basis of the evidence, some historians believe that he was not only very clever and very careful, but also very lucky.

Appendix IX

RIGID MOTION

The General Idea of a Rigid Motion.

In Chapters 5 and 12 we have defined congruence in a number of different ways, dealing with various kinds of figures. The complete list looks like this:

(1) $\overline{AB} \cong \overline{CD}$ if the two segments have the same length, that is, if $AB = CD$.

(2) $\angle A \cong \angle B$ if the two angles have the same measure, that is, if $m\angle A = m\angle B$.

(3) $\Delta ABC \cong \Delta DEF$ if, under the correspondence $ABC \longleftrightarrow DEF$, every two corresponding sides are congruent and every two corresponding angles are congruent.

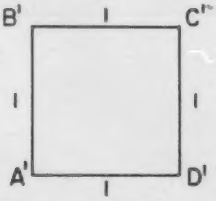
(4) Two circles are congruent if they have the same radius.

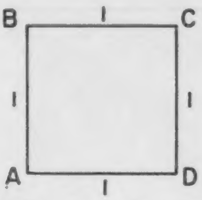
(5) Two circular arcs $\widehat{AB}$ and $\widehat{CD}$ are congruent if the circles that contain them are congruent and the two arcs have the same degree measure.

The intuitive idea of congruence is the same in all five of these cases. In each case, two cardboard figures are congruent if one of them can be moved so as to coincide with the other.

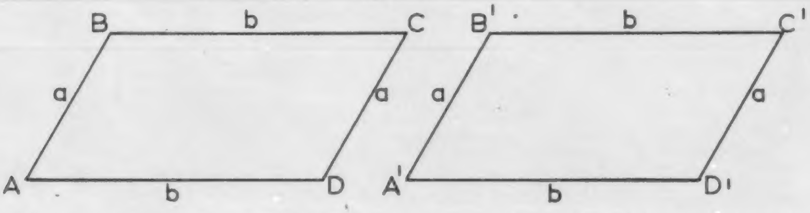
At the beginning of our study of congruence, the scheme used in Chapters 5 and 12 is the easiest and probably the best. It is a pity, however, to have five different special ways of describing the same basic idea in five special cases. And, in a way, it is a pity for this basic idea to be limited to these five special cases.

For example, as a matter of common sense, it is plain that two squares, each of edge 1, must be congruent in some valid sense:





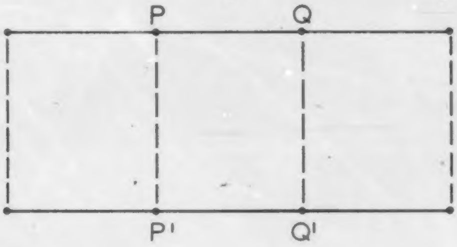
The same ought to be true for parallelograms, if corresponding sides and angles are congruent, like this:



It is plain, however, that none of our five special definitions of congruence applies to either of these cases.

In this appendix, we shall explain the idea of rigid motion. This idea is defined in exactly the same way, regardless of the type of figure to which we happen to be applying it. We shall show that for segments, angles, triangles, circles and arcs it means exactly the same thing as congruence. Finally, we will show that the squares and parallelograms in the figures above can be made to coincide by rigid motion. Thus, first, the idea of congruence will be unified, and second, the range of its application will be extended.

Before we give the general definition of a rigid motion, let us look at some simple examples. Consider two opposite sides of a rectangle, like this:



The vertical sides are dotted, because we will not be especially concerned with them. For each point P, Q, ..., of the top edge let us drop a perpendicular to the bottom edge; and let the foot of the perpendicular be P', Q' ... . Under this procedure, to each point of the top edge there corresponds exactly one point of the bottom edge. And conversely, to each point of the bottom edge there corresponds exactly one point of the top edge. We can't write down all of the matching pairs $P \longleftrightarrow P'$, $Q \longleftrightarrow Q'$, ..., because there are infinitely many of them. We can, however, give a general rule, explaining what is to correspond to what; and in fact, this is what we have done. Usually, we will write down a typical pair

$$P \longleftrightarrow P',$$

and explain the rule by which the pairs are to be formed.

Notice that the idea of a one-to-one correspondence is exactly the same in this case as it was when we were using it for triangles in Chapter 5. The only difference is that if we are matching up the vertices of two triangles, we can write down all of the matching pairs, because there are only three of them. ($ABC \longleftrightarrow DEF$ means that $A \longleftrightarrow D$, $B \longleftrightarrow E$ and $C \longleftrightarrow F$.) At present, we are talking about exactly the same sort of things, only there are too many of them to write down.

It is very easy to check that if P and Q are any two points of the top edge, and P' and Q' are the corresponding points of the bottom edge, then

$$PQ = P'Q'.$$

This is true because the segments $\overline{PQ}$ and $\overline{P'Q'}$ are opposite sides of a rectangle. We express this fact by saying that the correspondence $P \longleftrightarrow P'$ preserves distances.

The correspondence that we have just set up is our first and simplest example of a rigid motion. To be exact:

Definition: Given two figures F and F', a rigid motion between F and F' is a one-to-one correspondence

$$P \longleftrightarrow P'$$

between the points of F and the points of F', preserving distances.

If the correspondence $P \longleftrightarrow P'$ is a rigid motion between F and F', then we shall write

$$F \approx F'.$$

This notation is like the notation $\triangle ABC \cong \triangle A'B'C'$ for congruences between triangles. We can read $F \approx F'$ as "F is isometric to F'." ("Isometric" means "equal measure.")

Problem Set IX-1

1. Consider triangles $\triangle ABC$ and $\triangle A'B'C'$, and suppose that $$\triangle ABC \cong \triangle A'B'C'.$$ Let F be the set consisting of the vertices of the first triangle, and let F' be the set consisting of the vertices of the second triangle. Show that there is a rigid motion $$F \approx F'.$$

2. Let F be the set consisting of the vertices of a square of edge 1, and let F' be the set consisting of the vertices of another square of edge 1, as in the figure at the beginning of this Appendix. Show that there is a rigid motion $$F \approx F'.$$ (First, you have to explain what corresponds to what, and second you have to verify that distances are preserved.)

3. Do the same for the vertices of the two parallelograms in the figure at the start of this Appendix.

4. Show that if F consists of three collinear points, and F' consists of three non-collinear points, then there is no rigid motion between F and F'. (What you will have to do is to assume that such a rigid motion exists, and then show that this assumption leads to a contradiction.)

5. Show that there is never a rigid motion between two segments of different lengths.

6. Show that there is never a rigid motion between a line and an angle. (Hint: Apply Problem 4.)

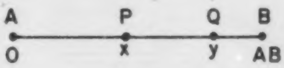
7. Show that given any two rays, there is a rigid motion between them. (Hint: Use the Point Plotting Theorem.)
8. Show that there is never a rigid motion between two circles of different radius.

Rigid Motion of Segments.

Theorem IX-1. If $AB = CD$, then there is a rigid motion $\overline{AB} \approx \overline{CD}$.

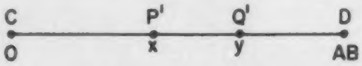
Proof: First, we need to set up a correspondence $P \longleftrightarrow P'$ between $\overline{AB}$ and $\overline{CD}$. Then, we need to check that distances are preserved.

By the Ruler Postulate, the points of the line $\overleftrightarrow{AB}$ can be given coordinates in such a way that A has coordinate zero and B the positive coordinate AB.



In the figure, we have shown typical points P, Q with their coordinates x and y.

In the same way, the points of CD can be given coordinates:



Notice that D has the coordinate AB, because $CD = AB$.

It is now plain what rule we should use to set up the correspondence

$$P \longleftrightarrow P'$$

between the points of $\overline{AB}$ and the points of $\overline{CD}$. The rule is that P corresponds to P' if P and P' have the same coordinate. (In particular, $A \longleftrightarrow C$ because A and C have coordinate zero, and $B \longleftrightarrow D$ because B and D have coordinate AB.)

It is easy to see that this correspondence is a rigid motion. If $P \longleftrightarrow P'$ and $Q \longleftrightarrow Q'$, and the coordinates are x and y, as in the figure, then $PQ = P'Q'$, because

$$PQ = |y - x| = P'Q'.$$

We therefore have a rigid motion

$$\overline{AB} \approx \overline{CD},$$

and the theorem is proved.

Notice that this rigid motion between the two segments is completely described if we explain how the end-points are to be matched up. We therefore will call it *the rigid motion induced by the correspondence*

$$A \longleftrightarrow C$$

$$B \longleftrightarrow D.$$

Theorem IX-2. If there is a rigid motion $\overline{AB} \approx \overline{CD}$ between two segments, then $AB = CD$.

The proof is easy. (This theorem was Problem 5 in the previous Problem Set.)

Problem Set IX-2

1. Show that there is another rigid motion between the congruent segments $\overline{AB}$ and $\overline{CD}$, induced by the correspondence

$$A \longleftrightarrow D$$

$$B \longleftrightarrow C.$$

2. Show that there are two rigid motions between a segment and itself. (One of these, of course, if the identity correspondence $P \longleftrightarrow P'$, under which every point corresponds to itself; this is a rigid motion because $PQ = PQ$ for every P and Q.)

Rigid Motion of Rays, Angles and Triangles.

Theorem IX-3. Given any two rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$, there is a rigid motion

$$\overrightarrow{AB} \approx \overrightarrow{CD}.$$

The proof of this theorem is quite similar to that of Theorem IX-1, and the details are left to the reader.

Theorem IX-4. If $\angle ABC \cong \angle DEF$, then there is a rigid motion

$$\angle ABC \approx \angle DEF$$

between these two angles.

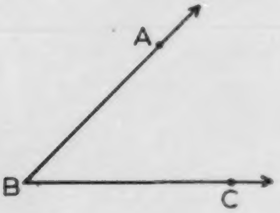
Proof: We know that there are rigid motions

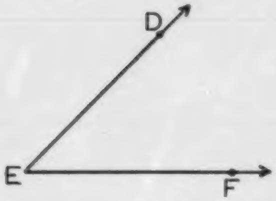
$$\overrightarrow{BA} \approx \overrightarrow{ED}$$

and

$$\overrightarrow{BC} \approx \overrightarrow{EF}$$

between the rays which form the sides of the two angles.



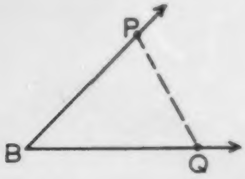


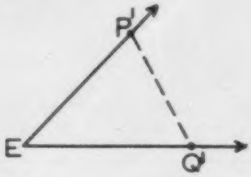
Let us agree that two points P and P' (or Q and Q') are to correspond to one another if they correspond under one of these two rigid motions. This gives us a one-to-one correspondence between the two angles. What we need to show is that this correspondence preserves distances.

Suppose that we have given two points P, Q of ABC and the corresponding points P', Q' of DEF. If P and Q are on the same side of ABC, then obviously

$$P'Q' = PQ,$$

because distances are preserved on each of the rays that form $\angle ABC$. Suppose, then, that P and Q are on different sides of $\angle ABC$, so that P' and Q' are on different sides of $\angle DEF$, like this:





By the S.A.S. Postulate, we have

$$\triangle PBQ \cong \triangle P'EQ'.$$

Therefore $PQ = P'Q'$, which was to be proved.

Next, we need to prove the analogous theorem for triangles:

Theorem IX-5. If

$$\triangle ABC \cong \triangle A'B'C',$$

then there is a rigid motion

$$\triangle ABC \approx \triangle A'B'C',$$

under which A, B and C correspond to A', B', and C'.

Proof: First, we shall set up a one-to-one correspondence between the points of $\triangle ABC$ and the points of $\triangle A'B'C'$. We have given a one-to-one correspondence

$$ABC \longleftrightarrow A'B'C'$$

for the vertices. By Theorem VIII-1 this gives us the induced rigid motions

$$\overline{AB} \approx \overline{A'B'},$$

$$\overline{AC} \approx \overline{A'C'}$$

and

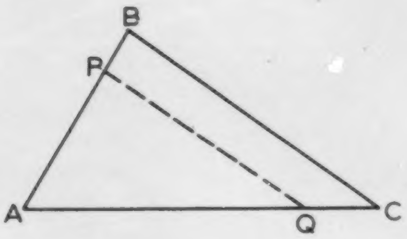
$$\overline{BC} \approx \overline{B'C'}$$

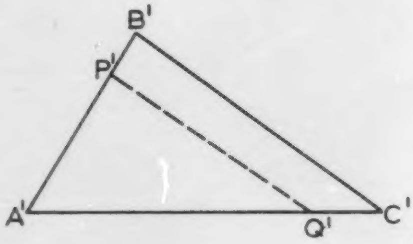
between the sides of the triangles. These three rigid motions, taken together, give us a one-to-one correspondence $P \longleftrightarrow P'$ between the points of the two trignales. We need to show that this correspondence preserves distances.

If P and Q are on the same side of the triangle, then we know already that

$$P'Q' = PQ.$$

Suppose, then, that P and Q are on different sides, say, $\overline{AB}$ and $\overline{AC}$, like this:





We know that

$$AP = A'P',$$

because $\overline{AB} \approx \overline{A'B'}$ is a rigid motion. For the same reason,

$$AQ = A'Q',$$

and $\angle A = \angle A'$, because $\triangle ABC \cong \triangle A'B'C'$. By the S.A.S. Postulate,

$$\triangle PAQ \cong \triangle P'A'Q'.$$

Therefore,

$$PQ = P'Q',$$

which was to be proved.

Notice that while the figure does not show the case $P = B$, the proof takes care of this case. The proof is more important than the figure, anyway.

Problem Set IX-3

1. Let

   $$ABC \longrightarrow A'B'C'$$

   be a rigid motion, and suppose that A, B, and C are collinear. Show that if B is between A and C, then B' is between A' and C'.

2. Given a rigid motion

   $$F \approx F'.$$

   Let A and B be points of F, and suppose that F contains the segment $\overline{AB}$. Show that F' contains the segments $A'B'$.

3. Given a rigid motion

   $$F \approx F'.$$

   Snow that if F is convex, then so also is F'.

4. Given a rigid motion $F \approx F'$. Show that if F is a ray, then so also is F'.

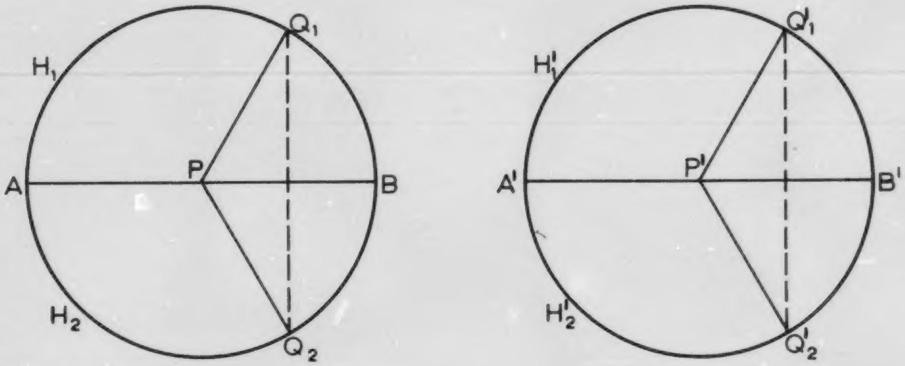
5. Show that there is no rigid motion between a segment and a circular arc (no matter how short both of them may be).

Rigid Motion of Circles and Arcs.

Theorem IX-6. Let C and C' be circles of the same radius r. Then, there is a rigid motion

$$C \approx C'$$

between C and C'.



Proof: Let the centers of the circles be P and P'. Let $\overline{AB}$ be a diameter of the first circle, and let $\overline{A'B'}$ be a diameter of the second. Let H_1 and H_2 be the half-planes determined by the line $\overleftrightarrow{AB}$; and let H'_1 and H'_2 be the half-planes determined by the line $\overleftrightarrow{A'B'}$.

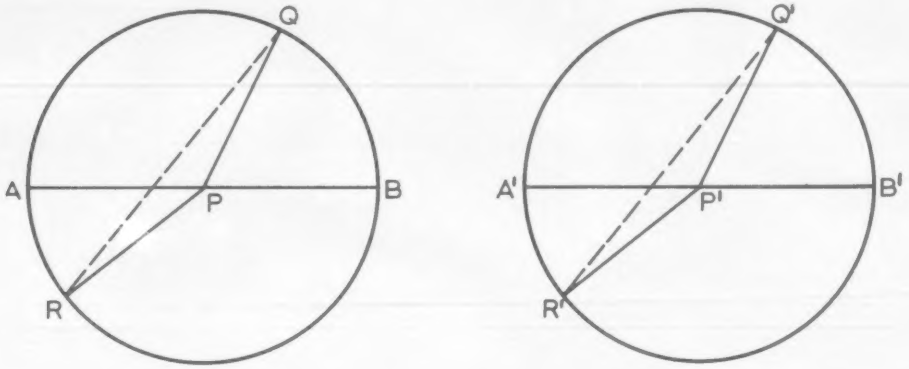
We can now set up our one-to-one correspondence $Q \longleftrightarrow Q'$, in the following way: (1) Let A' and B' correspond to A and B, respectively. (2) If Q_1 is a point of C, lying in H_1, let Q'_1 be the point of C', lying in H'_1, such that

$$\angle Q'_1P'B' \cong \angle Q_1PB.$$

(3) If Q_2 is a point of C, lying in H_2, let Q'_2 be the point of C_2, lying in H'_2, such that

$$\angle Q'_2P'B' = \angle Q_2PB.$$

We need to check that this correspondence preserves distances.



Thus, for every two points Q, R of C, we must have

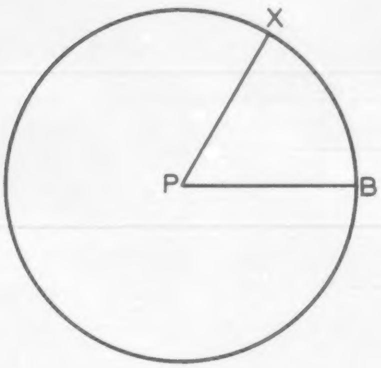
$$Q'R' = QR.$$

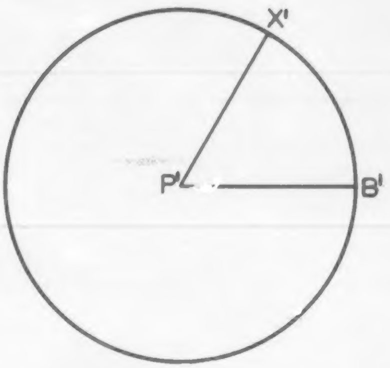
If Q and R are the end-points of a diameter, then so are Q' and R', and $Q'R' = QR = 2r$. Otherwise, we always have $\Delta QPR \cong \Delta Q'P'R'$, so that $Q'R' = QR$. (Proof? There are two cases to consider, according as B is in the interior or the exterior of $\angle QPR$.)

You should prove the following two theorems for yourself. They are not hard, once we have gone this far.

Theorem IX-7. Let C and C' be circles with the same radius, as in Theorem IX-6.

Let $\angle XPB$ and $\angle X'P'B'$ be congruent central angles of C and C'. respectively.





Then a rigid motion $C \approx C'$ can be chosen in such a way that $B \longleftrightarrow B'$, $X \longleftrightarrow X'$, and $\widehat{BX} \approx \widehat{B'X'}$.

Theorem IX-8. Given any two congruent arcs, there is a rigid motion between them. The proof is left to the reader.

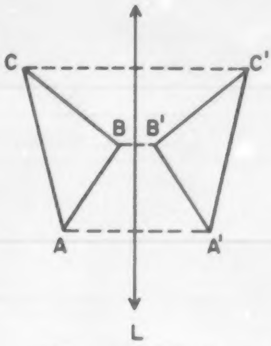
Reflections.

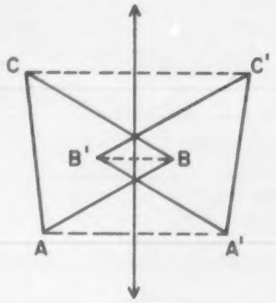
The definition of rigid motion given in Section VIII-1 is a perfectly good mathematical definition, but we might claim that from an intuitive viewpoint it does not convey any idea of "motion". We will devote this section to showing how a plane figure can be "moved" into coincidence with any isometric figure in the same plane.

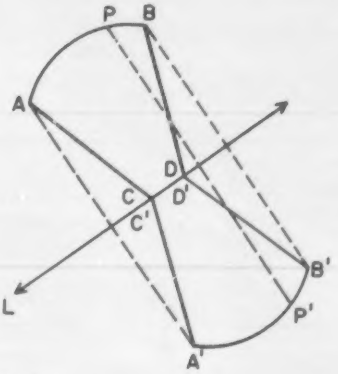
Throughout this section all figures will be considered as lying in a fixed plane.

Definitions. A one-to-one correspondence between two figures is a *reflection* if there is a line L, such that for any pair of corresponding points P and P', either (1) $P = P'$ and lies on L or (2) L is the perpendicular bisector of $\overline{PP'}$.
L is called the *axis of reflection*, and each figure is said to be the *reflection*, or the *image*, of the other figure in L.

In the pictures below are shown some examples of reflections of simple figures.

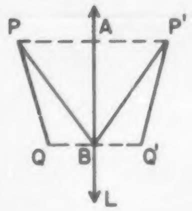




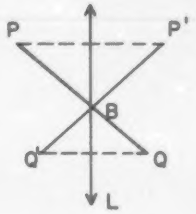


Theorem IX-9. A reflection is a rigid motion.

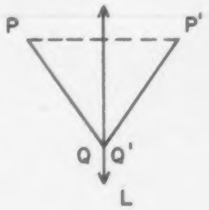
Proof: We must show that if P and Q are any two points, and P' and Q' their images in a line L, then $PQ = P'Q'$. There are four cases to consider.



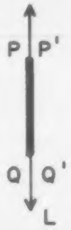
Case (1)



Case (2)



Case (3)



Case (4)

Case 1. P and Q are on the same side of L. Let $\overline{PP'}$ intersect L at A and $\overline{QQ'}$ intersect L at B. By the definition of reflection $\overline{PP'} \perp L$ and $PA = P'A$, and $\overline{QQ'} \perp L$ and $QB = Q'B$. Hence $\Delta PAB \cong \Delta P'AB$, and $PB = P'B$, $\angle PBA \cong \angle P'BA$. By subtraction, $\angle PBQ \cong \angle PBQ'$. We then have (by S.A.S.) $\Delta PBQ \cong \Delta P'BQ'$, and so $PQ = P'Q'$.

Case 2. The proof is the same, except that in proving $\angle PBQ \cong \angle PBQ'$ we add angle measures instead of subtracting.

Case 3. Q is on L. Then $Q = Q'$ and $PQ = P'Q'$ since Q is on the perpendicular bisector of $\overline{PP'}$. The case P on L and Q not on L is just the same.

Case 4. P and Q both on L. Since $P = P'$ and $Q = Q'$ we certainly have $PQ = P'Q'$.

Starting with a figure F we can reflect it in some line to get a figure F_1, F_1 can be reflected in some line to get a figure F_2, and so on. If we end up with a figure F' after n such steps we shall say that F has been carried into F' by a chain of n reflections.

Corollary IX-9-1. A chain of reflections carrying F into F' determines a rigid motion between F and F'.

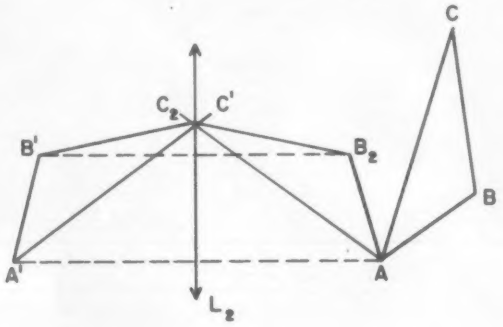
Coming back to our opening discussion in this section, a reflection can be thought of as a physical motion, obtained by rotating the whole plane through 180° about the axis of reflection. The above corollary says that a certain type of rigid motion, namely, those obtainable as a chain of reflections, can be given in a physical interpretation. What we shall now show is that every rigid motion if of this type.

The proof will be given in two stages, the first stage involving only a very simple figure. For convenience we will use the notation $F \mid F'$ if F and F' are reflections of each other in some axis.

Theorem IX-10. Let A, B, C, A', B', C' be six points such that $AB = A'B'$, $AC = A'C'$, $BC = B'C'$. Then there is a chain of at most three reflections that carries A, B, C into A', B', C'.

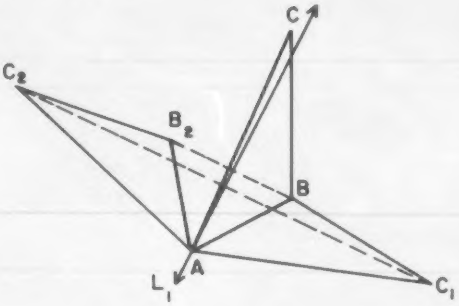
Proof:

Step 1.



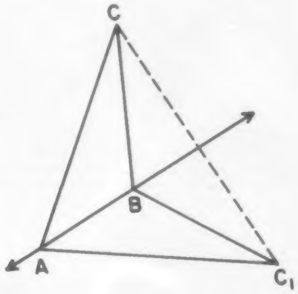
Let L_2 be the perpendicular bisector of $\overline{AA'}$, and let B_2 and C_2 be the reflections of B' and C' in L_2. Then $A, B_2, C_2 \mid A', B', C'$.

Step 2.



Let L_1 be the perpendicular bisector of $\overline{BB_2}$. Since $AB = A'B'$ and since by Theorem IX-9, $A'B' = AB_2$, it follows that $AB = AB_2$. Therefore A lies on L_1 and is its own image in the reflection in L. Thus, the image of A, B_2, C_2 in L_1 is A, B, C_1.

Step 3.



By arguments similar to the one above, we see that $AC = AC$ and $BC = BC_1$. Hence, $\overleftrightarrow{AB}$ is the perpendicular bisector of $\frac{1}{CC_1}$, and the image A, B, C_1 in $\overleftrightarrow{AB}$ is A, B, C.

We thus have,

$$A, B, C \mid A, B, C_1 \mid A, B_2, C_2 \mid A', B', C',$$

as was desired.

Any one or two of the three steps may be unnecessary if the pair of points we are working on (A in step 1, B in step 2, C in step 3) happen to coincide.

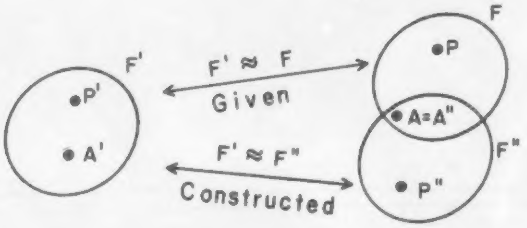
We are now ready for the final stage of the proof.

Theorem IX-11. Any rigid motion is the result of a chain of at most three reflections.

Proof: We are given a rigid motion $F \approx F'$. Let A, B, C be three non-collinear points in F, and A', B', C' the corresponding points in F'.

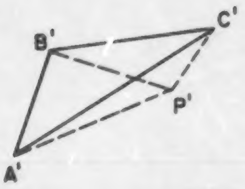
(If all points of F are collinear a separate, but simpler, proof is needed. The details of this are left to the student.)

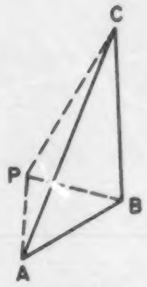
By Theorem IX-10, we can pass from A', B', C' to A, B, C by a chain of at most three reflections. By corollary IX-9-1, this chain determines a rigid motion $F' \approx F''$, and by the construction of the reflections we have $A'' = A$, $B'' = B$ and $C'' = C$. Schematically the situation is something like this:



We shall show that for every point P of F, we have $P'' = P$. This will show that F'' coincides with F, and that the given rigid motion $F \approx F'$ is identical with the one determined by the chain of reflections.

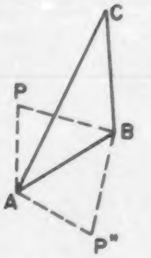
Let us consider, then, any point P of F, its corresponding point P' in F' determined by the rigid motion $F \approx F'$, and the point P'' in F'' determined from P' by the chain of reflections. We recall that $A'' = A$, $B'' = B$, $C'' = C$.





Since all our relationships are rigid motions, we have $AP'' = A'P' = AP$. Similarly, $BP'' = BP$ and $CP'' = CP$. From the first two of these, and $AB = AB$, we get that $\triangle ABP \cong \triangle ABP''$, and so $\angle BAP = \angle BAP''$. If P and P'' are on the same side of $\overleftrightarrow{AB}$ then by the Angle Construction Theorem $\overrightarrow{AP} = \overrightarrow{AP''}$, and since $AP = AP''$ it follows from the Point Plotting Theorem that $P = P''$, which is what we wanted to prove.

Suppose then that P and P'' lie on opposite sides of AB.

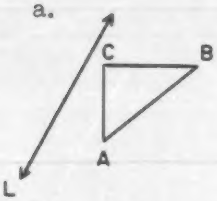


Since $PA = P''A$ and $PB = P''B$ it follows that A and B lie on the perpendicular bisector of $\overline{PP''}$. Since $PC = P''C$, C also lies on this line, contrary to the choice of A, B, and C as non-collinear. Hence, this case does not arise, and we are left with $P = P''$, thus proving the theorem.

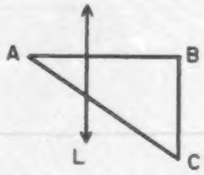
Problem Set IX-5

1. In each of the following construct, with any instruments you find convenient, the image of the given figure in the line L.

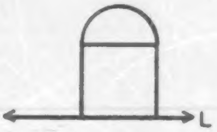
a.



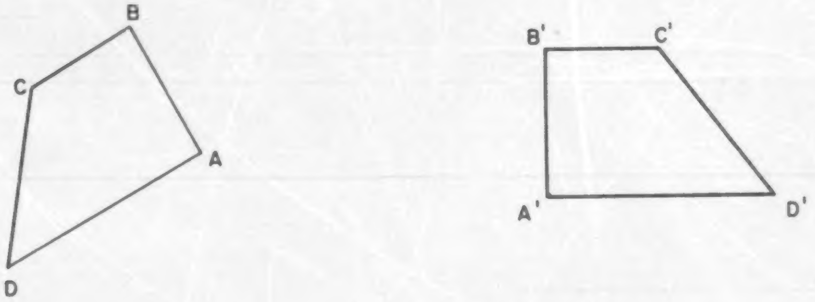
b.



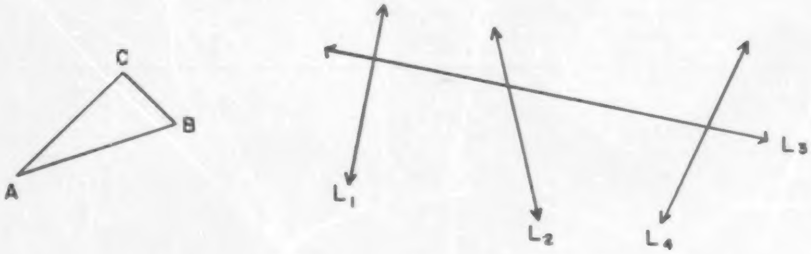
c.



2. Find a chain of three or fewer reflections that will carry ABCD into A'B'C'D'.



3. a. Carry ΔABC through the chain of four reflections in the axes L_1, L_2, L_3, L_4.



b. Find a shorter chain that will give the same rigid motion.

Definitions: A figure is symmetric if it is its own image in some axis. Such an axis is called an axis of symmetry of the figure.

4. Show that an isosceles triangle is symmetric. What is the axis?

5. A figure may have more than one axis of symmetry. How many do each of the following figures have?

   a. A rhombus.

   b. A rectangle.

   c. A square.

   d. An equilateral triangle.

   e. A circle.

6. The rigid motion defined by a chain of two reflections in parallel axes has the property that if $P \longleftrightarrow P'$ then $\overline{PP'}$ has a fixed length (twice the distance between the axes) and direction (perpendicular to the axes). Prove this. Such a motion is called a translation.

7. The rigid motion defined by a chain of two reflections in axes which intersect at Q has the property that if $P \longleftrightarrow P'$, then $\angle PQP'$ has a fixed measure (twice the measure of the acute angle between the axes). Prove this.

8. Show how by using the results of Problems 6 and 7 the Fundamental Theorem IX-11 can be restated in the following form:

   Any rigid motion in a plane is either a reflection, a translation, a rotation, a translation followed by a reflection, or a rotation followed by a reflection.

Appendix X

TRIGONOMETRY

Trigonometric Functions.

The elementary study of trigonometry is based on the following theorem.

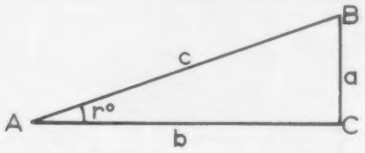
Theorem X-1. If an acute angle of one right triangle is congruent to an acute angle of another right triangle, then the two triangles are similar.

Proof: In $\triangle ABC$ and $\triangle A'B'C'$ let $\angle C$ and $\angle C'$ be right angles and let $m\angle A = m\angle A'$. Then $\triangle ABC \sim \triangle A'B'C'$ by A.A. Similarity Theorem 7-6.

We apply this theorem as follows: Let r be any number between 0 and 90, and let $\triangle ABC$ be a right triangle with $m\angle C = 90$ and $M\angle A = r$. For convenience set

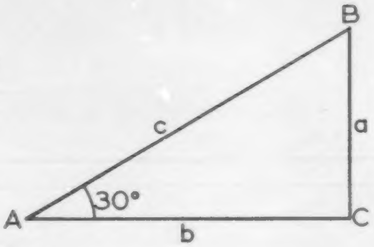
$$AB = c, \quad AC = b, \quad BC = a.$$

(The Pythagorean Theorem then tells us that $c^2 = a^2 + b^2$.)



If we consider another such triangle $\triangle A'B'C'$ with $m\angle C' = 90$ and $m\angle A' = r$, we get three corresponding numbers a', b', c', which would generally be different from a, b, c. However, we have $(a, c) \underset{p}{=} (a', c')$. By alternation we can then write $(a, a') \underset{p}{=} (c, c')$ and its constant of proportionality is $\frac{a}{c}$.

Thus, the number $\frac{a}{c}$ does not depend on the particular triangle we use, but only on the measure r of the acute angle. The value of this number is called the sine of r°, written $\sin r^\circ$ for short. The reason we specify that we are using degree measure is that in more advanced aspects of trigonometry a different measure of angle, radian measure, is common.



Let us see what we can say about sin 30°. We know from Theorem 11-9 that in this case if $c = 1$, then $a = \frac{1}{2}$. Hence, $\sin 30^\circ = \frac{a}{c} = \frac{1}{2}$.

It is evident that the number $\frac{b}{c}$ can be treated in the same way as $\frac{a}{c}$. The number $\frac{b}{c}$ is called the cosine of r°, written $\cos r^\circ$. From the Phythagorean Theorem, we see that if $a = \frac{1}{2}$ and $c = 1$, then $b = \sqrt{\frac{3}{2}}$. Hence, $\cos 30^\circ = \sqrt{\frac{3}{2}}$.

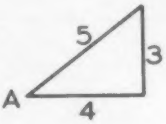
Of the four other possible quotients of the three sides of the triangle, we shall use only one, $\frac{a}{b}$. This is called the tangent of r°, written $\tan r^\circ$. We see that $\tan 30^\circ = \frac{1}{\sqrt{3}}$. (This use of the word "tangent" has only an unimportant historical connection with its use with relation to a line and a circle.)

These three quantities are called trigonometric functions.

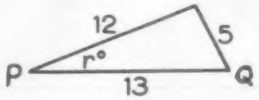
Problem Set X-1

1. In each of the following give the required information in terms of the indicated lengths of the sides.

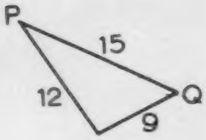
   a. $\sin A = ?$, $\cos A = ?$, $\tan A = ?$.



   b. $\sin r^\circ = ?$, $\cos P = ?$, $\tan P = ?$.

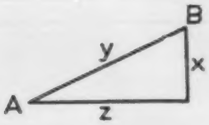


c. $\sin P = ?$, $\cos P = ?$, $\tan Q = ?$.



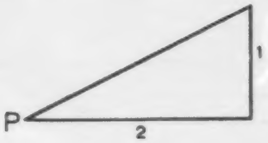
d. $\sin A = ?$, $\sin B = ?$,

$\tan A = ?$, $\tan B = ?$.

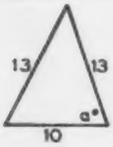


2. In each of the following, find the correct numerical value for x.

a. $\cos P = x$.



b. $\tan a^\circ = x$.



3. Find: $\sin 60^\circ$, $\cos 60^\circ$, $\tan 60^\circ$.

4. Find: $\sin 45^\circ$, $\cos 45^\circ$, $\tan 45^\circ$.

5. By making careful drawings with ruler and protractor determine by measuring

a. $\sin 20^\circ$, $\cos 20^\circ$, $\tan 20^\circ$;

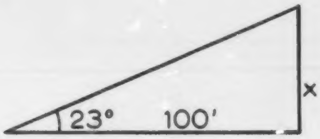
b. $\sin 53^\circ$, $\cos 53^\circ$, $\tan 53^\circ$.

Trigonometric Tables and Applications.

Although the trigonometric functions can be computed exactly for a few angles, such as 30°, 60° and 45°, in most cases, we have to be content with approximate values. These can be worked out by various advanced methods and at the end of this Appendix, we give a table of the values of the three trigonometric functions correct to three decimal places.

Having a "trig table", and a device for measuring angles, such as a surveyor's transit (or strings and a protractor) one can solve various practical problems.

Example X-1. From a point 100 feet from the base of a flag pole the angle between the horizontal and a line to the top of the pole is found to be 23°. Let x be the height of the pole. Then



$$\frac{x}{100} = \tan 23^\circ = .425.$$

Hence, $x = 42.5$ feet. An angle like the one used in this example is frequently called the angle of elevation of the object.

Example X-2. In a circle of radius 8 cm. a chord $\overline{AB}$ has length 10 cm. What is the measure of an angle inscribed in the major arc $\widehat{AB}$? We have $AC = 8$, $AQ = \frac{1}{2} \cdot 10 = 5$. Hence, $\sin(\angle ACQ) = \frac{5}{8} = .625$,

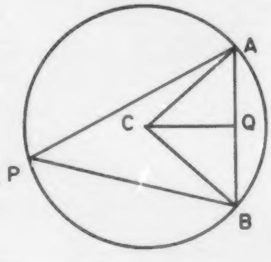
$m\angle ACQ = 39^\circ$,

$m(\text{minor arc } \widehat{AB}) = m\angle ACB =$

$2(m\angle ACQ) = 78^\circ$.

Hence, $m\angle APB = \frac{1}{2}m(\text{arc } \widehat{AB}) =$

39° to the nearest degree.



Problem Set X-2.

1. From the table find: $\sin 17^\circ$, $\cos 46^\circ$, $\tan 82^\circ$, $\cos 33^\circ$, $\sin 60^\circ$. Does the last value agree with the one found in Problem 3 of Set X-1?

2. From the table find x to the nearest degree in each of the following cases:

$$\cos x = .731, \quad \sin x = .390, \quad \tan x = .300$$

$$\sin x = .413, \quad \tan x = 2, \quad \cos x = \frac{1}{3}.$$

3. A hiker climbs for a half mile up a slope whose inclination is 17°. How much altitude does he gain?

4. When a six-foot pole casts a four-foot shadow, what is the angle of elevation of the sun?

5. An isosceles triangle has a base of 6 inches and an opposite angle of 30°. Find:

   a. The altitude of the triangle.

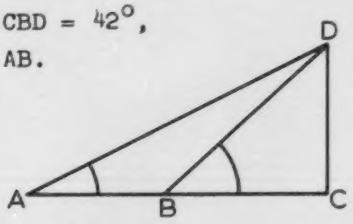
   b. The lengths of the altitudes to the equal sides.

   c. The angles these altitudes make with the base.

   d. The point of intersection of the altitudes.

6. A regular decagon (10 sides) is inscribed in a circle of radius 12. Find the length of a side, the apothem, and the area of the decagon.

7. Given, $m\angle A = 26^\circ$, $m\angle CBD = 42^\circ$, $BC = 50$; find AD and AB.



Relations Among the Trigonometric Functions.

Theorem X-2. For any acute $\angle A$; $\sin A < 1$, $\cos A < 1$.

Proof: In the right triangle ΔABC of Section X-1, $a < c$ and $b < c$. Dividing each of these inequalities by c gives

$$\frac{a}{c} < 1, \quad \frac{b}{c} < 1,$$

which is what we wanted to prove.

Theorem X-3. For any acute angle A,

$$\frac{\sin A}{\cos A} = \tan A, \quad \text{and} \quad (\sin A)^2 + (\cos A)^2 = 1.$$

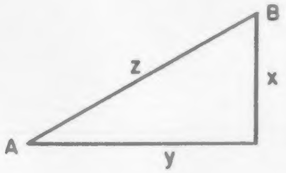
Proof:

$$\frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{b}{c}} = \tan A.$$

$$(\sin A)^2 + (\cos A)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2}$$

$$= \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1.$$

Theorem X-4. If A and B are complementary acute angles, then $\sin A = \cos B$, $\cos A = \sin B$, and $\tan A = \dfrac{1}{\tan B}$.



Proof: In the notation of the figure, we have

$$\sin A = \frac{x}{z} = \cos B,$$

$$\cos A = \frac{y}{z} = \sin B,$$

$$\tan A = \frac{x}{y} = \frac{1}{\frac{y}{x}} = \frac{1}{\tan B}.$$

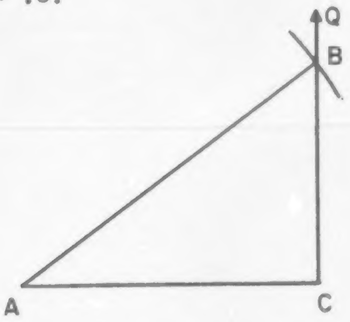
Problem Set X-3

Do the following problems without using the tables.

1. If $\sin A = \frac{1}{3}$ what is the value of $\cos A$? What is the value of $\tan A$? (Use Theorem X-3.)

2. With ruler and compass construct $\angle A$, if possible, in each of the following. You are allowed to use the results of earlier parts to simplify later on s.

a. $\cos A = .8$.



Solution: Take $\overline{AC}$ any convenient segment and construct $\overrightarrow{CQ} \perp \overline{AC}$. With center A and radius $\frac{AC}{.8}$ construct an arc intersecting $\overrightarrow{CQ}$ at B. Then $\cos(\angle BAC) = .8$.

b. $\cos A = \frac{2}{3}$.

c. $\cos A = \frac{3}{2}$.

d. $\sin A = .8$.

e. $\sin A = .7$.

f. $\tan A = \frac{2}{3}$.

g. $\tan A = \frac{3}{2}$.

Table of Trigonometric Ratios

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
0	0.000	1.000	0.000				
1	.018	1.000	.018	46	0.719	0.695	1.036
2	.035	0.999	.035	47	.731	.682	1.072
3	.052	.999	.052	48	.743	.669	1.111
4	.070	.998	.070	49	.755	.656	1.150
5	.087	.996	.088	50	.766	.643	1.192
6	.105	.995	.105	51	.777	.629	1.235
7	.122	.993	.123	52	.788	.616	1.280
8	.139	.990	.141	53	.799	.602	1.327
9	.156	.988	.158	54	.809	.588	1.376
10	.174	.985	.176	55	.819	.574	1.428
11	.191	.982	.194	56	.829	.559	1.483
12	.208	.978	.213	57	.839	.545	1.540
13	.225	.974	.231	58	.848	.530	1.600
14	.242	.970	.249	59	.857	.515	1.664
15	.259	.966	.268	60	.866	.500	1.732
16	.276	.961	.287	61	.875	.485	1.804
17	.292	.956	.306	62	.883	.470	1.881
18	.309	.951	.325	63	.891	.454	1.963
19	.326	.946	.344	64	.899	.438	2.050
20	.342	.940	.364	65	.906	.423	2.145
21	.358	.934	.384	66	.914	.407	2.246
22	.375	.927	.404	67	.921	.391	2.356
23	.391	.921	.425	68	.927	.375	2.475
24	.407	.914	.445	69	.934	.358	2.605
25	.423	.906	.466	70	.940	.342	2.747
26	.438	.899	.488	71	.946	.326	2.904
27	.454	.891	.510	72	.951	.309	3.078
28	.470	.883	.532	73	.956	.292	3.271
29	.485	.875	.554	74	.961	.276	3.487
30	.500	.866	.577	75	.966	.259	3.732
31	.515	.857	.601	76	.970	.242	4.011
32	.530	.848	.625	77	.974	.225	4.331
33	.545	.839	.649	78	.978	.208	4.705
34	.559	.829	.675	79	.982	.191	5.145
35	.574	.819	.700	80	.985	.174	5.671
36	.588	.809	.727	81	.988	.156	6.314
37	.602	.799	.754	82	.990	.139	7.115
38	.616	.788	.781	83	.993	.122	8.144
39	.629	.777	.810	84	.995	.105	9.514
40	.643	.766	.839	85	.996	.087	11.43
41	.658	.755	.869	86	.998	.070	14.30
42	.669	.743	.900	87	.999	.052	19.08
43	.682	.731	.933	88	.999	.035	28.64
44	.695	.719	.966	89	1.000	.018	57.29
45	.707	.707	1.000	90	1.000	.000	

Solutions to Appendix

Problem Set X-1

1. a. $\frac{3}{5}$, $\frac{4}{5}$, $\frac{3}{4}$.

   b. $\frac{5}{13}$, $\frac{12}{13}$, $\frac{5}{12}$.

   c. $\frac{3}{5}$, $\frac{4}{5}$, $\frac{4}{3}$.

   d. $\frac{x}{y}$, $\frac{z}{y}$, $\frac{x}{z}$, $\frac{z}{x}$.

2. a. $\frac{2}{\sqrt{5}}$, b. $\frac{12}{5}$.

3. $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\sqrt{3}$.

4. $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, 1.

5. a. .34, .94, .36.

   b. .80, .60, 1.33.

Problem Set X-2

2. 43°, 23°, 17°, 24°, 63°, 71°.

3. $\sin 17^\circ = \frac{x}{\frac{1}{2} \cdot 5280}$ $x = .292 \cdot 2640 = 771$ feet.

4. $\tan x = \frac{6}{4} = 1.5$. $x = 56^\circ$.

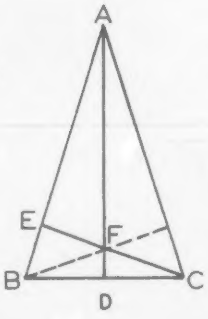
5. $m \angle A = 30$, $m \angle B = m \angle C = 75^\circ$.

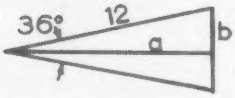
   a. $\frac{AD}{CD} = \tan C$. $AD = 3.732 \cdot 3 = 11.196$.

   b. $\frac{CE}{CB} = \sin B$. $CE = .966 \cdot 6 = 5.796$.

   c. $m \angle ECB = 90^\circ - m \angle B = 15^\circ$.

   d. $\frac{DF}{CD} = \tan 15^\circ$. $DF = .268 \cdot 3 = .804$.





6. $\sin 18^\circ = \frac{b}{12}$,

$b = 3.71$, $2b = \underline{7.42}$

$\cos 18^\circ = \frac{a}{12}$,

$a = \underline{11.41}$.

$\text{area} = \frac{1}{2} \cdot 10 \cdot 7.42 \cdot 11.41 = 423.$

7. $\tan 42^\circ = \frac{CD}{50}$, $CD = 45.0$.

$\tan 26^\circ = \frac{45}{AC}$, $AC = 92.2$, $AB = \underline{42.2}$.

$\sin 26^\circ = \frac{45}{AD}$, $AD = \underline{103}$.

Problem Set X-3

1. $(\sin A)^2 + (\cos A)^2 = 1$, $\frac{1}{9} + (\cos A)^2 = 1$,

$\cos A = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$.

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{1}{3}}{\frac{2\sqrt{2}}{3}} = \frac{1}{2\sqrt{2}}.$$

2. (c) is impossible.

(d) A here is congruent to B of part (a).

(g) A here is the complement of the A of part (f).

Appendix XI

VECTORS IN SPACE

In Chapter 10 our work dealt solely with vectors in a plane. However, all that we did there can be extended to space, and in this appendix, we summarize, without proof, the extensions of our results to three dimensions. (Definitions, notations, and theorems which are not repeated here are the same in two and three dimensions.)

Definitions. A *vector* in space is an ordered triple of real numbers [a, b, c]. The numbers a, b, c are called the *components* of the vector.

Definition. The ordered triple [0, 0, 0] is called the *zero vector*.

Definition. If $\vec{u} = [a, b, c]$ and h is a scalar, the vector [ha, hb, hc] is called the *product of the vector* $\vec{u}$ *and the scalar* h.

Definition. If $\vec{u} = [a, b, c]$, the vector [-a, -b, -c] is called the *opposite* of u.

Definition. If $\vec{u} = [a, b, c]$ the number $\sqrt{a^2 + b^2 + c^2}$ is called the *magnitude* or *length* of $\vec{u}$.

Definition. Two vectors are equal if and only if they have the same components.

Definition. If $\vec{u} = [a, b, c]$ and $\vec{v} = [d, e, f]$, the vector [a + d, b + e, c + f] is called the *sum* of $\vec{u}$ and $\vec{v}$.

Properties 1-11 hold equally well for vectors in a plane and vectors in space.

Definition. Three vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$ are said to be coplanar if there exists a plane to which $\overleftrightarrow{AB}$, $\overleftrightarrow{CE}$, $\overleftrightarrow{EF}$ are all parallel.

Theorem XI-1. If $\vec{u}$, $\vec{v}$, $\vec{w}$ are three non-zero and non-coplanar vectors, and if z is any vector, then there exist scalars p, q, r such that

$$\vec{z} = \overrightarrow{pu} + \overrightarrow{qv} + \overrightarrow{rw}.$$

Theorem XI-2. If $\vec{u}$, $\vec{v}$, $\vec{w}$ are three non-zero and non-coplanar vectors, and if $p_1, q_1, r_1, p_2, q_2, r_2$ are scalars such that

$$p_1\vec{u} + q_1\vec{v} + r_1\vec{w} = p_2\vec{u} + q_2\vec{v} + r_2\vec{w},$$

then

$$p_1 = p_2,\ q_1 = q_2,\ r_1 = r_2.$$

Definition. If $\vec{u} = [a, b, c]$ and $\vec{v} = [d, e, f]$, the number $ad + be + cf$ is called the scalar product of $\vec{u}$ and $\vec{v}$.

The properties of the scalar product are the same in two and three dimensions.

Theorem XI-3. Two non-zero vectors are perpendicular if and only if their scalar product is zero.

Theorem XI-4. If $\vec{u}$ and $\vec{v}$ are non-zero vectors, the absolute value of their scalar product is equal to

(a) The length of $\vec{u}$ multiplied by the length of the projection of $\vec{v}$ on $\vec{u}$; or, equally well,

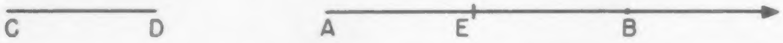
(b) The length of $\vec{v}$ multiplied by the length of the projection of $\vec{u}$ on $\vec{v}$.

APPENDIX XII

APPLICATIONS OF GEOMETRIC THEORY
to the
USE OF STRAIGHTEDGE AND COMPASSES
in
DRAWING PICTURES OF PLANE FIGURES

In your study of informal geometry in previous mathematics courses you may have learned how to use a straightedge and compasses in drawing pictures of plane geometric figures. The pictures below suggest some of the basic operations in the use of these tools. The comments give some indication of the related theory for the plane which may be applied to "prove the accuracy" of the picture.

1. Given a ray $\overrightarrow{AB}$ and a segment $\overline{CD}$, to draw the point E on $\overrightarrow{AB}$ such that $\overline{AE} \cong \overline{CD}$.

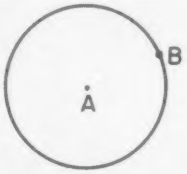


(The Point Plotting Theorem)

2. Given two distinct points, to draw the line which contains them.

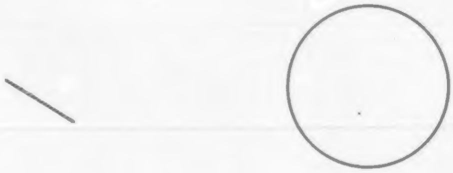
(Postulate 1)

3. Given two distinct points A and B , to draw the circle with center A and radius AB .



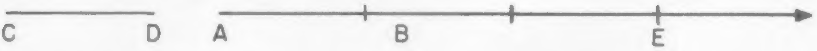
(Definition of Circle)

4. Given a segment and a point, to draw the circle with center at the point and radius equal to the measure of the segment.



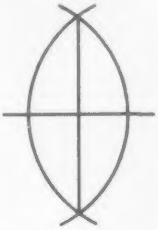
(Definition of Circle)

5. Given a ray $\overrightarrow{AB}$ and a segment $\overline{CD}$, to draw the point E on $\overrightarrow{AB}$ such that $\overline{AE} = 3 \cdot \overline{CD}$.



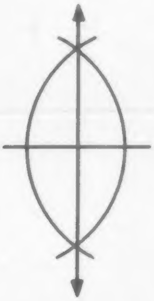
(Point Plotting Theorem)

6. To draw the midpoint of a given segment.

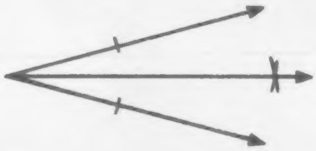


(An application of the Triangle Congruence Postulates)

7. To draw the perpendicular bisector of a given segment.

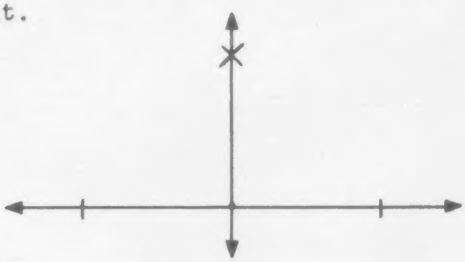


8. To draw the bisector of a given angle.



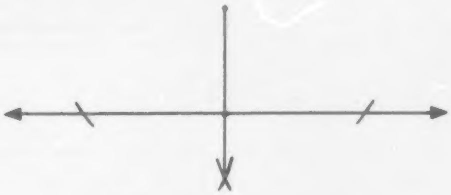
(An application of the S.S.S. Congruence Postulate)

9. To draw a line perpendicular to a given line at a given point on it.



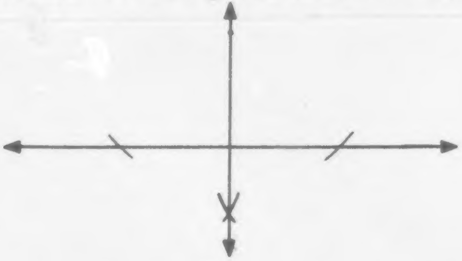
(An application of the S.S.S. Congruence Postulate)

10. Given a line and a point not on it, to draw the foot of the perpendicular from the point to the line.

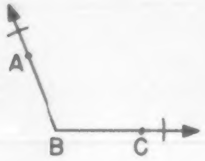


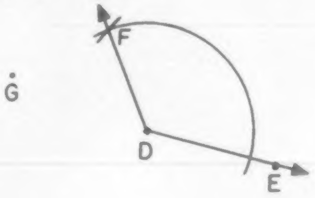
(An application of the S.S.S. Congruence Postulate)

11. Given a line and a point not on it, to draw the line which is perpendicular to the given line and contains the given point.



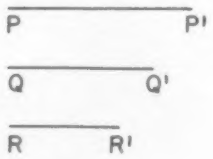
12. Given an angle $\angle ABC$, a ray $\overrightarrow{DE}$ and a point G not on $\overleftrightarrow{DE}$, draw the ray $\overrightarrow{DF}$ such that F and G are on the same side of $\overleftrightarrow{DE}$ and $\angle FDE \cong \angle ABC$.

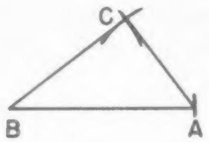




(An application of the S.S.S. Congruence Postulate)

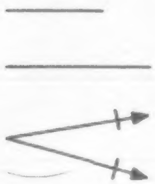
13. Given three segments $\overline{PP'}$, $\overline{QQ'}$, $\overline{RR'}$, such that the sum of the lengths of every pair of them exceeds the length of the third segment, draw a triangle ΔABC such that $AB = PP'$, $BC = QQ'$, $AC = RR'$.

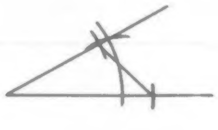




(Some algebra involving coordinates and equations may be used to show that the circles with centers at A and B , with $AB = PP'$, and radii QQ' and RR' , respectively, intersect in two points, one on each side of $\overleftrightarrow{AB}$.)

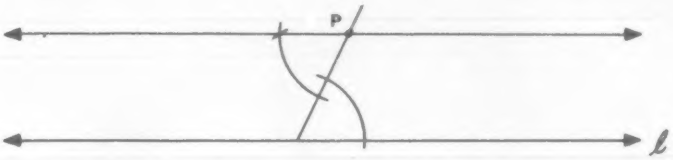
14. Given two segments and an angle, to draw a triangle having two sides and an included angle congruent to the given segments and angle.





(An application involving some of the incidence postulates and (12) above.)

15. Given a line ℓ a point P not on it, to draw the line through the given point and parallel to the given line.



(An application involving (12) above.)

Problem Set XII

The following are exercises in the use of straightedge and compasses:

1. Given a circle with center M and chord $\overline{RS}$ not containing M, to draw a chord $\overline{AB}$ through M and perpendicular to $\overline{RS}$.

   The chord $\overline{AB}$ __________ the chord $\overline{RS}$.
   The chord $\overline{AB}$ is a __________ of the circle.

2. Given a circle and a radius which joins its center to a point P on it, to draw a line perpendicular to the radius at P.

3. Given a circle with center P and chord $\overline{AB}$ not containing P, to draw the chord $\overline{AC}$ perpendicular to $\overline{AB}$, and also to draw the chord $\overline{BC}$.

   The chord $\overline{BC}$ __________ the point P.

4. Given a circle with center P and chord $\overline{CD}$ not containing P, to draw the perpendicular bisector of $\overline{CD}$.

   This bisector contains the point __________ .

5. Given a circle with center R and chords $\overline{MA}$ and $\overline{AC}$, to draw the intersection of the perpendicular bisectors of $\overline{MA}$ and $\overline{AC}$.

   This intersection is __________ .

6. Given three noncollinear points $A, B, C,$ to draw a circle which contains $A, B,$ and C.

   How many distinct circles contain these three points?

7. Given a triangle ABC, to draw the triangle whose vertices are the midpoints of ΔABC.

8. Given a convex quadrilateral $ABCD$ to draw the quadrilateral $A'B'C'D'$ where A', B', C', D' are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, respectively.

9. Given a line segment $\overline{AB}$, to draw the point C on $\overline{AB}$ such that $AC = 2 \cdot CB$.

10. Given two distinct points A and B, to draw a point C such that $AC = \sqrt{3} \cdot AB$.

11. Given two distinct points A and B, to draw an angle $\angle CBA$ such that $m \angle CBA = 60$.

12. Given two distinct points A and B, to draw a regular hexagon ABCDEF.

13. Given three noncollinear points, to draw a parallelogram having the given points as three of its vertices.

14. Given a triangle, to draw its medians. The medians intersect in the centroid of the triangle.

15. Given a triangle, to draw its angle bisectors. These angle bisectors intersect in the incenter of the triangle.

16. Given a triangle, to draw its altitudes. These altitudes intersect in the orthocenter of the triangle.

17. Given a triangle, to draw the perpendicular bisectors of its sides. These bisectors intersect in the circumcenter of the triangle.

18. Given a triangle, to draw its incircle.

19. Given a triangle ABC, to draw a triangle DEF similar to ABC with constant of proportionality $\frac{2}{3}$.

20. Given three noncollinear points, to draw an equilateral triangle containing the three points.

The Meaning and Use of Symbols

General.

$=$ $A = B$ can be read as "A equals B", "A is equal to B", "A equal B" (as in "Let $A = B$"), and possibly other ways appears. However, we should not use the symbol, $=$, in such forms as "A and B are $=$"; its proper use is between two expressions. If two expressions are connected by "$=$" it is to be understood that these two expressions are names of the same mathematical object, in our case either a real number or a point set.

$\neq$ "Not equal to". $A \neq B$ means that A and B do not represent the same object. The same variations and cautions apply to the use of $\neq$ as to the use of $=$.

$\{a:$ property$\}$. The set of all elements a each of which has the stated property. (The "set-builder" notation).

Algebraic.

$+ \quad \cdot \quad - \quad \div$ These familiar algebraic symbols for operating with real numbers need no comment. The basic postulates about them are presented in Appendix II.

$< \quad > \quad \underline{<} \quad \underline{>}$ Like $=$, these can be read in various ways in sentences, and $A < B$ may stand for the underlined part of "If A is less than B", "Let A be less than B", "A less than B implies", etc. Similarly for the other three symbols, read "greater than", "less than or equal to", "greater than or equal to". These inequalities apply only to real numbers. Their properties are mentioned briefly in Section 3-2, and in more detail in Section 3-3.

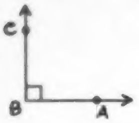
$|A|$ "Absolute value of A". Discussed in Sections 3-2 and 8-3.

(x, y) The coordinates of a point (an ordered pair of real numbers); also used as the name of the point.

$\underset{p}{=}$ Proportionality. "$(a, b, c) \underset{p}{=} (d, e, f)$" is read "$a, b, c$" are proportional to "$d, e, f$".

Geometric.

Point Sets.	A single letter may stand for any suitable described point set. Thus, we may speak of a point P, a line m, a halfplane $\mathcal{H}$, a circle C, an angle x, a segment b, etc.
$\overleftrightarrow{AB}$	The line containing the two points A and B.
$\overline{AB}$	The segment having A and B as endpoints.
$\overrightarrow{AB}$.	The ray with A as its endpoint and containing point B.
$\angle ABC$	The angle having B as vertex and $\overrightarrow{BA}$ and $\overrightarrow{BC}$ as sides.
$\triangle ABC$	The triangle having A, B, C as vertices.



$\angle ABC$ is a right angle.

$\angle A\text{-}BC\text{-}D$.	The dihedral angle having line $\overleftrightarrow{BC}$ as edge and with faces containing A and D.
$(\overrightarrow{A,B})$	Directed segment whose origin is A and whose terminus is B. Read "directed segment AB".
$\vec{u}$	The vector $\vec{u}$, read "u vector" or "the vector u".
$[a,b]$	The vector whose components are a and b. Note that a and b form an ordered pair and in general $[a,b]$ is not the same as $[b,a]$.
$\lvert\vec{u}\rvert$	The magnitude, or length, of $\vec{u}$
$\overrightarrow{PQ}$	The vector whose components are the same as those of directed segment $(\overrightarrow{P,Q})$. Read "PQ vector" or "the vector PQ".
$\vec{0}$	The zero vector; that is the vector whose components are both zero.
$\vec{u}\cdot\vec{v}$	The scalar product of the vectors $\vec{u}$ and $\vec{v}$. Read "u dot v".

$\angle V\text{-}Q_1Q_2\ldots Q_n$	The polyhedral angle whose vertex is V and whose edges are $\overrightarrow{VQ_1}$, $\overrightarrow{VQ_2}$, ..., $\overrightarrow{VQ_n}$.
$\overset{\frown}{AB}$	An arc of a circle with endpoints A and B. Read "arc AB".
$\overset{\frown}{AXB}$	The arc of a circle with endpoints A and B and containing the point X. Read "arc AXB".
$m\overset{\frown}{AXB}$	The degree measure of the arc whose endpoints are A and B and which includes the point X.
π	The greek letter pi, used here and many other places to denote the quotient of the circumference of a circle divided by its diameter. The number π is the same for all circles.

Real Numbers.

AB	The positive number which is the distance between the two points A and B, and also the length of the segment $\overline{AB}$.
$m\angle ABC$.	The real number between 0 and 180 which is the degree measure of $\angle ABC$.
PQ(relative to $\{A, A'\}$)	The measure of the segment $\overline{PQ}$ with respect to the unit-pair $\{A, A'\}$.
$m_{\overline{AB}}$	The slope of segment $\overline{AB}$.
$m_{\overrightarrow{AB}}$	The slope of ray $\overrightarrow{AB}$.
$m_{\overleftrightarrow{AB}}$	The slope of line $\overleftrightarrow{AB}$.

Relations.

$a \longleftrightarrow b$,	a is matched with b.
$\cong$	Congruence. $A \cong B$ is read "A is congruent to B", but with the same possible variations and restrictions as $A = B$. In the text A and B may be segments, angles, or triangles.

$\perp$.	Perpendicular. $A \perp B$ is read "A is perpendicular to B", but with the same comment as for $\cong$. A and B may be either two lines, or subsets of lines (rays or segments).
$F \approx F'$	F is isometric to F'. (Appendix IX).
$\parallel$.	Parallelism. We read "$p \parallel q$" as "line p is parallel to line q".
$\sim$.	Similarity. We read "$\Delta ABC \sim \Delta DEF$" as "triangle ABC is similar to triangle DEF".
$\doteq$	Equivalence for directed segments $(\overrightarrow{A, B}) \doteq (\overrightarrow{C, D})$ is read "directed segment $(\overrightarrow{A, B})$ is equivalent to directed segment $(\overrightarrow{C, D})$".

The Greek Alphabet

A	α	alpha	a	(a)	N	ν	nu	n	(n)
B	β	beta	b	(b)	Ξ	ξ	xi	x	(ks)
Γ	γ	gamma	g	(g)	O	ο	omicron	o	(o)
Δ	δ	delta	d	(d)	Π	π	pi	p	(p)
E	ε	epsilon	e	(e)	P	ρ	rho	r, rh	(r)
Z	ζ	zeta	z	(z)	Σ	σ	sigma	s	(s)
H	η	eta	e	(a)	T	τ	tau	t	(t)
Θ	θ	theta	th	(th)	ϒ	υ	upsilon	y, u	(u, oo)
I	ι	iota	i	(e)	Φ	φ	phi	ph	(f)
K	κ	kappa	k	(k)	X	χ	chi	ch	(k, K)
Λ	λ	lambda	l	(l)	Ψ	ψ	psi	ps	(ps)
M	μ	mu	m	(m)	Ω	ω	omega	o	(o)

LIST OF CHAPTERS AND POSTULATES

CHAPTER 8. Coordinates in a Plane

CHAPTER 9. Perpendicularity, Parallelism, and Coordinates in Space

POSTULATE 24. There is a unique plane which contains a given point and is perpendicular to a given line.

POSTULATE 25. Two lines which are perpendicular to the same plane are parallel.

CHAPTER 10. Directed Segments and Vectors

CHAPTER 11. Polygons and Polyhedrons

POSTULATE 26. If R is any given polygonal-region, there is a correspondence which associates to each polygonal-region in space a unique positive number, such that the number assigned to the given polygonal-region R is one.

POSTULATE 27. Suppose that the polygonal-region R is the union of two polygonal-regions R_1 and R_2 such that the intersection of R_1 and R_2 is contained in a union of a finite number of segments. Then, relative to a given unit-area, the area of R is the sum of the areas of R_1 and R_2 .

POSTULATE 28. If two triangles are congruent, then the respective triangular-regions consisting of the triangles and their interiors have the same area relative to any given unit-area.

POSTULATE 29. Given a unit-pair for measuring distance, the area of a rectangle relative to a unit-square is the product of the measures (relative to the given unit-pair) of any two consecutive sides of the rectangle.

CHAPTER 12. Circles and Spheres

POSTULATE 30. If $\widehat{AB}$ and $\widehat{BC}$ are arcs of the same circle having only the point B in common, and if their union is an arc $\widehat{AC}$, then $m\widehat{AB} + m\widehat{BC} = m\widehat{AC}$.

POSTULATE 31. The lengths of arcs in congruent circles are proportional to their degree measures.

LIST OF THEOREMS AND COROLLARIES

THEOREM 8-1. If P and Q are points on the same vertical line, then $PQ = |y_P - y_Q|$.

THEOREM 8-2. If P and Q are points on the same horizontal line, then $PQ = |x_P - x_Q|$.

THEOREM 8-3. Every vertical line is perpendicular to every horizontal line.

THEOREM 8-4. If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are two points in the xy-plane, then

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \; .$$

THEOREM 8-5. If P and Q are two points in the same vertical line, then the midpoint M of $\overline{PQ}$

$$M = \left(x_P \, , \, \frac{y_P + y_Q}{2}\right).$$

THEOREM 8-6. If P and Q are two points on the same horizontal line, then the midpoint M of $\overline{PQ}$ is the point

$$M = \left(\frac{x_P + x_Q}{2} \, , \, y_P\right) \; .$$

THEOREM 8-7. If P and Q are distinct points on a line which is neither vertical nor horizontal, then the midpoint M of $\overline{PQ}$ is the point

$$M = \left(\frac{x_P + x_Q}{2} \, , \, \frac{x_P + y_Q}{2}\right) \; .$$

THEOREM 8-8. If $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ are any two distinct points in a plane, then the midpoint M of $\overline{PQ}$ is the point

$$M = \left(\frac{x_2 + x_1}{2} \, , \, \frac{y_2 + y_1}{2}\right) \; .$$

THEOREM 8-9. Let a be any real number. Then the set of all points in the xy-plane each of which has x-coordinate a is a vertical line.

THEOREM 8-10. Let b be any real number. The set of all points in the xy-plane with y-coordinate b is a horizontal line.

THEOREM 8-11. If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are any two points, then
$\overleftrightarrow{P_1P_2} = \{(x,y) : x = x_1 + k(x_2 - x_1),\ y = y_1 + k(y_2 - y_1),\ k \text{ is real}\}$.

THEOREM 8-12. If a, b, c, d are real numbers such that b and d are not both zero and if
$S = \{(x,y) : x = a + bk,\ y = c + dk,\ k \text{ is real}\}$, then S is a line.

THEOREM 8-13. The slope of a non-vertical line p is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

, where $\overline{P_1P_2}$ is any segment of p and $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$.

THEOREM 8-14. If p is the line through (x_1,y_1) with slope $m = \frac{f}{g}$, then

1. $p = \{(x,y) : x = x_1 + kg\ ,\ y = y_1 + kf\ ,\ k \text{ is real}\}$ and
2. $p = \{(x,y) : x = x_1 + k\ ,\ y = y_1 + km\ ,\ k \text{ is real}\}$.

THEOREM 8-15. Two non-vertical lines are parallel if and only if their slopes are equal.

COROLLARY 8-15. Three points A, B, C are collinear if and only if $m_{\overline{AB}} = m_{\overline{BC}}$, or they lie on a vertical line.

THEOREM 8-16. If $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ and if PQ is an oblique line, then

$$\overleftrightarrow{PQ} = \left\{ (x,y) : \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \right\}.$$

COROLLARY 8-16-2. If p is the line which passes through $P(x_1,y_1)$ with slope m , then

$$p = \{(x,y) : y - y_1 = m(x - x_1)\}.$$

THEOREM 8-17. Two non-vertical lines are perpendicular if and only if the product of their slopes is -1 .

THEOREM 8-18. A quadrilateral is a parallelogram if each of its sides is congruent to the side opposite it.

THEOREM 8-19. A quadrilateral is a parallelogram if and only if each angle is congruent to the angle opposite it.

THEOREM 8-20. A quadrilateral is a rectangle if and only if it is equiangular.

THEOREM 8-21. A quadrilateral is a rhombus if and only if it is equilateral.

THEOREM 8-22. A line segment which joins the midpoints of two sides of a triangle is parallel to the third side and its length is half the length of the third side.

THEOREM 8-23. Given quadrilateral ABCD with $A = (0,0)$, $B = (a,0)$, $D = (b,c)$, then ABCD is a parallelogram is and only if $C = (a + b,c)$.

COROLLARY 8-23-1. If the coordinates of the vertices of a parallelogram are $A = (0,0)$, $B = (a,0)$, $C = (a + b,c)$, and $D = (b,c)$, then the parallelogram is a rectangle if and only if $b = 0$.

COROLLARY 8-23-2. If the coordinates of the vertices of a parallelogram are $A = (0,0)$, $B = (a,0)$, $C = (a + b,c)$ and $D = (b,c)$ where $a > 0$, then the parallelogram is a rhombus if and only if $a = \sqrt{b^2 + c^2}$.

THEOREM 8-24. A quadrilateral is a parallelogram if and only if the diagonals bisect each other.

THEORME 8-25. A parallelogram is a rectangle if and only if the diagonals are congruent.

THEOREM 8-26. A parallelogram is a rhombus if and only if the diagonals are perpendicular.

THEOREM 8 27. A parallelogram is a rhombus if and only if a diagonal bisects one of its angles.

THEOREM 8-28. The set of all points in a plane which are equidistant from two given points in the plane is the perpendicular bisector of the segment joining the given points.

COROLLARY 8-28-1. The perpendicular bisectors of the sides of a triangle are concurrent at a point equidistant from the vertices of the triangle.

THEOREM 8-29. The set of all points in the interior of an angle which are equidistant from the lines which contain the sides of the angle is the interior of the midray of the angle.

COROLLARY 8-29-1. The lines which contain the angle bisectors of the angles of a triangle are concurrent at a point equidistant from the sides of the triangle.

THEOREM 9-1. The plane which is perpendicular to a given line at a point contains every line which is perpendicular to the given line at that point.

THEOREM 9-2. If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane determined by the two lines.

THEOREM 9-3. There is a quique line which is perpendicular to a given plane at a given point in the plane.

THEOREM 9-4. If a plane intersects one of two distinct parallel lines in a point, it intersects the other line in a point also.

THEOREM 9-5. If aplane is parallel to one of two parallel lines, it is also parallel to the other.

THEOREM 9-6. If a plane intersects each of two distinct parallel planes, the intersections are two distinct parallel lines.

THEOREM 9-7. If a line intersects one of two distinct parallel planes in a single point, it intersects the other plane in a single point also.

THEOREM 9-8. If aline is parallel to one of two parallel planes, it is parallel to the other also.

THEOREM 9-9. Two planes which are perpendicular to the same line are parallel.

THEOREM 9-10. If a line is perpendicular to one of two distinct parallel planes it is perpendicular to the other also.

THEOREM 9-11. If a plane is perpendicular to one of two distinct parallel lines, it is perpendicular to the other line also.

THEOREM 9-12. If two lines are parallel to a third line, they are parallel to each other.

THEOREM 9-13. Given a plane and a point not in the plane, there is a unique line which passes through the point and is perpendicular to the plane.

THEOREM 9-14. There is a unique plane parallel to a given plane through a given point.

THEOREM 9-15. If two planes are each parallel to a third plane, they are parallel to each other.

THEOREM 9-16. The shortest segment joining a point to a plane not containing the point is the segment perpendicular to the given plane.

THEOREM 9-17. All segments which are perpendicular to each of two distinct parallel planes and have their endpoints in the planes have the same length.

THEOREM 9-18. The set of all points which are equidistant from the endpoints of a given segment is the plane which contains the midpoint of the segment and is perpendicular to the line which contains the segment.

THEOREM 9-19. Any two plane angles of a dihedral angle are congruent.

THEOREM 9-20. If a line is perpendicular to a plane, then any plane containing this line is perpendicular to the given plane.

THEOREM 9-21. If two planes are perpendicular, then any line in one of the planes which is perpendicular to their line of intersection is perpendicular to the other plane.

THEOREM 9-22. If two planes are perpendicular, then any line perpendicular to one of the planes at a point on their line of intersection lies in the other plane.

THEOREM 9-23. If two intersecting planes are each perpendicular to a third plane, then their line of intersection is perpendicular to this plane.

THEOREM 9-24. If P_1 and P_2 are points on a line parallel to the x-axis, then $P_1P_2 = |x_1 - x_2|$, where x_1 and x_2 are the x-coordinates of P_1 and P_2 , respectively.

THEOREM 9-25. If P_1 and P_2 are points on a line parallel to the y-axis, then $P_1P_2 = |y_1 - y_2|$, where y_1 and y_2 are the y-coordinates of P_1 and P_2 , respectively.

THEOREM 9-26. If P_1 and P_2 are points on a line parallel to the z-axis, then $P_1P_2 = |z_1 - z_2|$, where z_1 and z_2 are the z-coordinates of P_1 and P_2 respectively.

THEOREM 9-27. The distance between the points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ is given by

$$P_2P_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} .$$

THEOREM 9-28. If $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ are any two distinct points, then for every value of k the point whose coordinates are

$$x = x_1 + k(x_2 - x_1)$$
$$y = y_1 + k(y_2 - y_1)$$
$$z = z_1 + k(z_2 - z_1)$$

lies on $\overleftrightarrow{P_1P_2}$ and, conversely, to every point on $\overleftrightarrow{P_1P_2}$ there corresponds a unique value of k such that these equations give the coordinates of the point.

THEOREM 9-29. Every plane has an equation of the form $ax + by + cz = d$, where one or more of the numbers a, b, c is different from zero; and every equation of this form is an equation of a plane.

THEOREM 10-1. There is one and only one directed segment which is equivalent to a given directed segment and has its origin at a given point.

THEOREM 10-2. Two directed segments $(\overrightarrow{P_1,P_2})$ and $(\overrightarrow{P_3,P_4})$ are equivalent if and only if they have the same components.

THEOREM 10-3. If P_1 and P_2 have coordinates (x_1,y_1) and (x_2,y_2) , respectively, the length of any directed segment equivalent to $(\overrightarrow{P_1,P_2})$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$

THEOREM 10-4. If the coordinates of P_1 and P_2 are (x_1,y_1) and (x_2,y_2) , respectively, then the components of the directed segment $(\overrightarrow{P_1,P_3})$ which is k times the directed segment $(\overrightarrow{P_1,P_2})$ are $k(x_2 - x_1)$ and $k(y_2 - y_1)$.

THEOREM 10-5. The components of $(\overrightarrow{P_1,P_2}) + (\overrightarrow{P_3,P_4})$ are the sums of the corresponding components of $(\overrightarrow{P_1,P_2})$ and $(\overrightarrow{P_3,P_4})$.

THEOREM 10-6. $(\overrightarrow{P_1,P_2}) + (\overrightarrow{P_3,P_4})$ and $(\overrightarrow{P_3,P_4}) + (\overrightarrow{P_1,P_2})$ are equivalent directed segments.

THEOREM 10-7. If $(\overrightarrow{P_1,P_2}) \doteq (\overrightarrow{Q_1,Q_2})$ and $(\overrightarrow{P_3,P_4}) \doteq (\overrightarrow{Q_3,Q_4})$ then $(\overrightarrow{P_1,P_2}) + (\overrightarrow{P_3,P_4}) \doteq (\overrightarrow{Q_1,Q_2}) + (\overrightarrow{Q_3,Q_4})$.

THEOREM 10-8. If $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$ are parallel vectors, then $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ are parallel.

THEOREM 10-9. If $\vec{u}$ and $\vec{v}$ are parallel vectors, then

$$\vec{v} = k\vec{u}$$

where

$$k = \frac{|\vec{v}|}{|\vec{u}|} .$$

THEOREM 10-10. The sum of the vector $\overrightarrow{P_1P_2}$ and the vector $\overrightarrow{P_3P_4}$ is the vector $\overrightarrow{P_1X}$ where X is the unique point such that $\overrightarrow{P_2X} = \overrightarrow{P_3P_4}$.

THEOREM 10-11. If $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are two non-zero vectors which are not parallel and if $\overrightarrow{OP}$ is any vector in the plane OAB , then there exist scalars h and k such that

$$\overrightarrow{OP} = h\overrightarrow{OA} + k\overrightarrow{OB} .$$

THEOREM 10-12. If $\vec{u}$ and $\vec{v}$ are non-zero, non-parallel vectors, and if x, y, z, w are scalars such that

$$x\vec{u} + y\vec{v} = z\vec{u} + w\vec{v} ,$$

then

$$x = z \text{ and } y = w .$$

THEOREM 10-13. The midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

THEOREM 10-14. The segment joining the midpoints of two sides of a triangle is parallel to the third side and the length of the segment is one half the length of the third side.

THEOREM 10-15. A quadrilateral is a parallelogram if and only if its diagonals bisect each other.

THEOREM 10-16. Two non-zero vectors are perpendicular if and only if the sum of the products of their respective components is zero.

THEOREM 11-1. The sum of the measures of the angles of a convex polygon of n sides is $(n - 2) \cdot 180$.

COROLLARY 11-1-1. The measure of each angle of a regular polygon of n sides is

$$\frac{(n - 2)180}{n}, \text{ or } 180 - \frac{360}{n} .$$

THEOREM 11-2. For any convex polygon of n sides, the sum of the measures of exterior angles, one at each vertex of the polygon, is 360 .

COROLLARY 11-2-1. The measure of each exterior angle of a regular polygon of n sides is $\frac{360}{n}$.

THEOREM 11-3. The area of a right triangle is one half the product of the lengths of its two legs.

THEOREM 11-4. The area of a triangle is one-half the product of any base and the altitude to that base.

COROLLARY 11-4-1. The area A of an equilateral triangle whose side has length s is given by:

$$A = \frac{\sqrt{3}}{4} s^2 .$$

THEOREM 11-5. The area of a rhombus is one half the product of the lengths of the diagonals.

COROLLARY 11-5-1. The area A of the square whose diagonal has length d is given by

$$A = \frac{1}{2}d^2 .$$

THEOREM 11-6. The area of a parallelogram is the product of any base and the altitude to that base.

THEOREM 11-7. The area of a trapezoid is one-half the product of its altitude and the sum of its bases.

COROLLARY 11-7-1. The area of a trapezoid is equal to the product of its altitude and the length of its median.

THEOREM 11-8. Consider a set of two or more triangles.

(a) If the bases of all the triangles are equal, then the areas of the triangles are proportional to the corresponding altitudes.

(b) If the altitudes of all the triangles are equal, then the areas of the triangles are proportional to the corresponding bases.

(c) If the areas of all the triangles are equal, then the bases of the triangles are inversely proportional to the corresponding altitudes.

THEOREM 11-9. Consider a set of two or more parallelograms.

(a) If the bases of all the parallelograms are equal, then the areas of the parallelograms are proportional to the corresponding altitudes.

(b) If the altitudes of all the parallelograms are equal, then the areas of the parallelograms are proportional to the corresponding bases.

(c) If the areas of all the parallelograms are equal, then the bases of the parallelograms are inversely proportional to the corresponding altitudes.

THEOREM 11-10. Every similarity between triangles has the property that the measures of the three sides and any altitude of the one triangle are proportional to the measures of the corresponding sides and the corresponding altitude of the other triangle.

THEOREM 11-11. Every similarity between triangles has the property that the areas of the triangles are proportional to the squares of the lengths of any pair of corresponding sides.

THEOREM 11-12. Every similarity between convex polygons with n sides has the property that the lengths of the n sides and the perimeter of one polygon are proportional to the lengths of the corresponding sides and the perimeter of the other polygon.

THEOREM 11-13. Every similarity between convex polygons with n sides has the property that the areas of the polygonal-regions (consisting of the polygons and their interiors, respectively) are proportional to the squares of the lengths of any pair of corresponding sides.

THEOREM 11-14. The bisectors of the interior angles of a regular convex polygon of n sides intersect at a point.

THEOREM 11-15. Every central triangle of a regular polygon is isosceles and is congruent to every other central triangle.

THEOREM 11-16. The area of a regular polygon is one-half the product of the apothem and the perimeter of the polygon.

THEOREM 11-17. The sum of the measures of any two face angles of a trihedral angle is greater than the measure of the third face angle.

THEOREM 11-18. The sum of the measures of all the face angles of any polyhedral angle is less than 360 .

THEOREM 11-19. There are no more than five types of regular polyhedrons.

THEOREM 11-20. The lateral area of a prism is equal to the product of the length of a lateral edge and the perimeter of a right-section.

COROLLARY 11-20-1. The lateral area of a right prism is the product of the length of a lateral edge and the perimeter of a base.

THEOREM 11-21. Let a triangular pyramid be given.

(a) Every cross-section of the pyramid is a triangle similar to the boundary of the base.

(b) If the distance from the vertex of the pyramid to the plane containing the cross-section is k and if the altitude of the pyramid is h , then the area of the cross-section and the area of the base are proportional to the numbers k^2 and h^2 .

THEOREM 12-1. The intersection of a sphere with a plane through its center is a circle whose center and radius are the same as those of the sphere.

THEOREM 12-2. The radii of a circle or congruent circles, or of a sphere or congruent spheres, are congruent.

THEOREM 12-3. The diameters of a circle or congruent circles, or of a sphere or congruent spheres, are congruent.

THEOREM 12-4. Given a line ℓ and a circle C in the same plane. Let P be the center of the circle, and let F be the foot of the perpendicular from P to the line.

(1) Every point of ℓ is outside C if and only if F is outside C.

(2) ℓ is a tangent to C if and only if F is on C.

(3) ℓ is a secant of C if and only if F is inside C.

COROLLARY 12-4-1. Given a circle and a coplanar line, the line is a tangent to the circle if and only if it is perpendicular to a radius of the circle at its outer end.

COROLLARY 12-4-2. A diameter of a circle bisects a non-diameter chord of the circle if and only if it is perpendicular to the chord.

COROLLARY 12-4-3. In the plane of a circle, the perpendicular bisector of a chord contains the center of the circle.

COROLLARY 12-4-4. If a line in the plane of a circle intersects the interior of the circle, then it intersects the circle in exactly two points.

THEOREM 12-5. Chords of congruent circles are congruent if and only if they are equidistant from the centers.

THEOREM 12-6. Given a plane m and a sphere S with center P. Let F be the foot of the perpendicular from P to m.

1. Every point of m is outside S if and only if F is outside S.
2. m is tangent to S if and only if F is on S.
3. m intersects S in a circle with center F if and only if F is inside S.

COROLLARY 12-6-1. A plane is tangent to a sphere if and only if it is perpendicular to a radius at its outer endpoint.

COROLLARY 12-6-2. A perpendicular from the center of a sphere to a chord of the sphere bisects the chord.

COROLLARY 12-6-3. The segment joining the center of a sphere to the midpoint of a chord is perpendicular to the chord.

THEOREM 12-7. The measure of an inscribed angle is half the measure of its intercepted arc.

COROLLARY 12-7-1. An angle inscribed in a semicircle is a right angle.

COROLLARY 12-7-2. Angles inscribed in the same arc are congruent.

COROLLARY 12-7-3. Congruent angles inscribed in congruent circles intercept congruent arcs.

THEOREM 12-8. In the same circle or in congruent circles, if two chords, not diameters, are congruent, then so are the associated minor arcs.

THEOREM 12-9. In the same circle or in congruent circles, if two arcs are congruent, then so are the associated chords.

THEOREM 12-10. The measure of a tangent-chord angle is one-half the measure of its intercepted arc.

THEOREM 12-11. The measure of an angle whose vertex is in the interior of a circle and whose sides are contained in two secants, is one-half the sum of the measures of the intercepted arcs.

THEOREM 12-12. The measure of a secant-secant angle, or a tangent-tangent angle or a secant-tangent angle is one-half the difference between the measures of the intercepted arcs.

THEOREM 12-13. The two tangent-segments to a circle from an external point are congruent, and form congruent angles with the line joining the external point to the center of the circle.

THEOREM 12-14. The product of the length of a secant-segment from a given exterior point and the length of its external secant-segment is constant for any secant containing the given point.

THEOREM 12-15. Given a tangent-segment $\overline{QT}$ to a circle at T and a secant through Q, intersecting the circle in points R and S. Then

$$QR \cdot QS = (QT)^2 .$$

THEOREM 12-16. If two chords of a circle intersect, the product of the lengths of the segments of one is equal to the product of the lengths of the segments of the other.

THEOREM 12-17. The quotient of the circumfercnce divided by the diameter, $\frac{C}{2r}$, is the same for all circles.

COROLLARY 12-17-1. The circumference of circles are proportional to their radii.

THEOREM 12-18. The area of a circle of radius r is πr^2 .

COROLLARY 12-18-1. The areas of two circles are proportional to the squares of their radii.

THEOREM 12-19. An arc of degree measure q contained in a circle whose radius is r has length L , where

$$L = \frac{\pi r}{180} \cdot q .$$

THEOREM 12-20. The area of a sector is half the product of its radius and the length of its arc.

THEOREM 12-21. The area of a sector of radius r and arc measure q is

$$\frac{\pi r^2}{360} \cdot q .$$

THEOREM 12-22. A triangle has one and only one circumscribed circle. The center of this circle is the intersection of the perpendicular bisectors of the sides of the triangle.

THEOREM 12-23. A triangle has one and only one inscribed circle. The center of this circle is the intersection of the midrays of the angles of the triangle.

INDEX

For precisely defined geometric terms the reference is to the formal definition. For other terms the reference is to an informal definition or to the most prominent discussion.

A.A. similarity theorem, 413
absolute value, 81 ex., 519
acute angle, 182
addition property,
    of equality, 235
    of order, 63
    of proportionality, 398
adjacent angles, 181
alternate interior angles, 319
alternation property of
    proportion, 399
altitude
    of a parallelogram, 748
    of a prism, 799
    of a pyramid, 803
    of a trapezoid, 593
    of a triangle, 427
"and", 534
angle(s), 145
    acute, 182
    adjacent, 181
    alternate interior, 319
    bisector of, 167
    central, 847
    complementary, 189
    congruent, 184
    consecutive interior, 319
    of a convex polygon, 213
    corresponding, 319
    dihedral, 215
    exterior of a polygon, 738
    exterior of a triangle, 292
    face, 789
    inscribed, 851
    intercepts an arc, 852
    interior of a polygon, 738
    interior of a triangle, 292
    linear pair of, 179
    measure of, 154
    obtuse, 182
    plane angle of a dihedral
        angle, 634
    polyhedral, 788
    of a quadrilateral, 204
    reflex, 143
    right, 182
    secant-secant, 862
    secant-tangent, 862
    side of, 145
    "straight", 144
    supplementary, 189
    tangent-chord, 861
angle(s), (con't.)
    tangent-tangent, 862
    of a triangle, 201
    trihedral, 789
    vertex of, 145
    vertical, 194
    "zero", 144
angle construction theorem, 160
antiparallel rays, 355
apothem of regular polygon, 779
arc(s)
    congruent, 855
    degree measure of, 849
    endpoints of, 848
    intercepted, 852
    length of, 900
    major, 848
    minor, 848
    of a sector, 901
    semicircle, 848
area
    of a circle, 895
    of equilateral triangle, 755
    lateral, of a prism 799
    of a parallelogram, 756
    of polygonal regions, 744
    of a rectangle, 748
    of a regular polygon, 786
    of a rhombus, 755
    of a sector of a circle, 901
    of a square, 749, 756
    of a trapezoid, 758
    of triangles, 753
area relations
    of congruent triangles, 747
    of parallelograms, 769
    of triangles, 767
A.S.A. postulate, 252
axioms, 10
base(s)
    of isosceles triangle, 277
    of a parallelogram, 748
    of a prism, 796
    of a pyramid, 803
    of a trapezoid, 593
base angles
    of isosceles triangle, 277
    of a trapezoid, 593
betweenness
    for points, 91
    for rays, 165

- betweenness-addition theorem, 240
- betweenness-angles theorem, 166
- betweenness-coordinates theorem, 109
- betweenness-distance theorem, 117
- bisection
  - of an angle, 167
  - of a segment, 92
- boundary of a polygonal region, 732
- center
  - of a circle, 819
  - of gravity, 771 ex.
  - of a regular polygon, 779
  - of a sphere, 820
- central angle of a circle, 847
- central triangle of a regular polygon, 779
- chess, 34
- chord, 821
- circle(s), 819
  - area of, 895
  - area of sector of, 901
  - center of, 819
  - central angle of, 847
  - chord of, 821
  - circumference of, 888
  - circumscribed, 905
  - concentric, 820
  - congruent, 822
  - diameter of, 821
  - exterior of, 829
  - great, 821
  - inscribed, 905
  - interior of, 829
  - major arc of, 848
  - minor arc of, 848
  - power of a point with respect to, 871
  - radius of, 819, 821
  - secant of, 821
  - sector of, 901
  - segment of, 903
  - tangent of, 830
  - tangent externally, 835
  - tangent internally, 835
- circular-region, 894
- circumference of a circle, 888
- circumscribed circles, 905
- circumscribed triangle, 905
- collinear, 40
  - in that order, 91
- common external tangent, 877 ex.
- common internal tangent, 877 ex.
- complement, 189
- components
  - of directed segments, 690
  - of vectors, 703
- composite condition, 533
- concentric, 820
- conclusion, 10
- concurrent lines, 597
- concurrent rays, 597
  - in that order, 166
- concurrent segments, 597
- conditionals, 244
- congruence between two convex polygons, 460
- congruence between two triangles, 228
- congruent
  - angles, 184
  - arcs, 855
  - chords, 834
  - circles, 822
  - polygons, 405
  - segments, 115
  - spheres, 822
  - triangles, 229
- consecutive interior angles, 319
- constant of proportionality, 393
- contrapositive, 328
  - property of, 329
- converse, 280
  - of Pythagorean theorem, 434
- convex polygon(s), 211
  - angles of, 213
  - consecutive angles of, 213
  - diagonals of, 212
  - interior of, 212
- convex polyhedron, 884
- convex set of points, 134
- coordinate planes, 643
- coordinate of a point, 76
- coordinate system, 76
  - in a plane, 509
  - in space, 641
  - on a line, 505
  - origin of, 76
  - unit point of, 76
- coordinates of a point
  - in a plane, 511
  - in space, 646
- coplanar, 44
- correspondence, one-to-one, 30
  - between triangles, 227
- corresponding angles, 319
- counter-example, 5
- counting numbers, 55
- cross-section of a prism, 798
- cube, 797
- decagon, 210

decahedron, 784
deductive reasoning, 10
definitions, 15
  circular, 34
  complete form, 241
  formal, 15
  if and only if form, 242
  in proofs, 241
degree, 154
degree measure of an arc, 849
diagonals of a convex
  polygon, 212
diameter, 821
dihedral angle(s), 215
  edge of, 215
  face of, 215
  measure of, 635
  plane angle of, 634
  right, 635
  vertical, 216
directed segment(s), 684
  equal, 685
  equivalent, 686
    properties of, 687
  length of, 685
  opposite of, 693
  product with a number, 693
  subtraction of, 701 ex.
  sum of, 697
  x-component of, 690
  y-component of, 690
displacement, 683
distance,
  between a point and a
    line, 376
  between a point and a
    plane, 628
  between two parallel
    lines, 354
  between two points, 522, 655
  measure of, 70
distance formula,
  in a plane, 522
  in space, 655
dodecagon, 210
dodecahedron, 784
"dot product", 718
edge of
  halfplane, 138
  polygonal-region, 734 ex.
  polyhedral angle, 788
  polyhedron, 783
empty set, 26
endpoints of an arc, 848
equal directed segments, 685
equal vectors, 704
equation(s), 538
  equivalent, 539
  intercept form, 571 ex.
  parametric, 546, 658
  of a plane, 663
  point-slope form, 569
  slope-intercept form, 571 ex.
  two-point form, 569
equiangular triangle, 277
equilateral triangle, 277
  area of, 755
equivalent directed
  segments, 686
equivalent equations, 539
Euler's theorem, 734 ex.
exterior
  of an angle, 176
  of a circle, 829
  of a sphere, 841
  of a triangle, 203
exterior angle of polygon, 738
exterior angle of triangle, 292
external secant segment, 870
externally tangent circles, 835
face angle of a
  polydehral angle, 789
faces
  of a polygonal-region, 734 ex.
  of a polyhedral angle, 789
  of a polyhedron, 783
foot of a perpendicular, 294
frustum of a pyramid, 805
geometrical applications
  of vectors, 714
grad, 153
graph, 514
great circle of a sphere, 821
greater than, 57
halfline, 137
halfplane, 138
  edge of, 138
halfspace, 139
heptagon, 210
heptahedron, 784
hexagon, 210
hexahedron, 784
horizontal lines, 510
hypotenuse, 366
hypotenuse-leg theorem, 367
hypothesis, 10
icosahedron, 784
identity correspondence, 33 ex.
if and only if form, 242
if-then form, 17
incidence relations, 35
  points and lines, 36
  points, lines, and planes, 42

indirect method of proof, 325
indirect reasoning, 12
inductive reasoning, 5
inequalities, 538
  in the same order, 370
initial point, 684
inscribed angle, 851
inscribed circle, 905
inscribed triangle, 905
integers, 56
  negative, 56
  positive, 55
intercept form of a linear equation, 571
intercepted arc, 852
interior
  of an angle, 175
  of a circle, 829
  of a convex polygon, 212
  of a polygonal-region, 732
  of a ray, 90
  of a segment, 90
  of a sphere, 841
  of a triangle, 202
interior angle
  of a polygon, 738
  of a triangle, 292
internally tangent circles, 835
intersect, 27
intersection of sets, 24, 534
inversely proportional, 766
inversion property of proportion, 399
isosceles trapezoid, 593
isosceles triangle, 277
  base of, 277
  base angles of, 277
  theorem, 275
  vertex of, 277
lateral area of a prism, 799
lateral edge of a prism, 797
lateral face of a prism, 797
lateral surface of a prism, 797
leg of a right triangle, 366
leg of a trapezoid, 593
length of an arc, 900
length of a segment, 114
length of a vector, 704
less than, 64
limit, 888
line(s)
  concurrent, 597
  coordinate system on, 505
  horizontal, 510
  intercept form of, 571 ex.
  opposite sides of, 138
line(s), (con't.)
  parallel, 316, 343
  parallel to a plane, 617
  parametric equations of, 546
  perpendicular, 183
  perpendicular to a plane, 610
  point-slope form of, 569
  projection of a point on, 428
  projection of a segment on, 428
  representation of, 35
  skew, 316
  slope of, 556
  slope-intercept form of, 571 ex.
  transversal, 317
  two-point form of, 569
  undefined, 33
  vertical, 510
linear pair, 179
Lobachevskian geometry, 340
locus, 538
logical equivalence, 329
logical system, 2
magnitude of a vector, 704
major arc, 848
measure of an angle, 154
measure of an arc, 849
measure of a dihedral angle, 635
measure of distance, 70
median of a trapezoid, 593, 758
median of a triangle, 289
midpoint of a segment, 91, 526, 550
midray, 167
mil, 153
minor arc, 848
multiplication property
  of equality, 236
  of order, 63
nonagon, 210
nonahedron, 784
non-Euclidean geometries, 339
null set, 29 ex.
numbers
  counting, 55
  inequality of, 57
  integers, 56
  irrational, 56
  natural, 55
  negative, 63
  order properties, 63
  positive, 63
  rational 56
  real, 56
obtuse angle, 182
octagon, 210
octahedron, 784

one-to-one correspondence, 30
   between triangles, 228
"or", 534
order,
   for real numbers, 63
   of collinear points, 91
ordered pair, 511
ordered triple, 646
origin, 76, 510, 684
origin and unit point theorem, 78
outer end of radius, 821
parallel lines, 316, 343
   distance between, 354
   properties of, 347
parallel postulate, 339
parallel rays, 355
parallel segments, 350
parallel vectors, 705
parallelepiped, 797
   rectangular, 797
parallelism, 315
   of a line to a plane, 617
   of two planes, 617
parallelogram, 351
   altitude of, 748
   area of, 756
   base of, 748
   properties of, 603
parameter, 546
parametric equations
   in a plane, 546
   in space, 658
pentagon, 210
pentahedron, 784
perpendicular, 183
   foot of, 294
   lines, 183
   planes, 635
   sets, 183
   vectors, 717
perpendicularity of a line and a plane, 610
pi, π , 889
plane(s),
   coordinate system in, 509
   equation of, 663
   parallel to a line, 617
   parallel to another plane, 617
   perpendicular, 635
   perpendicular to a line, 610
   representation of, 44
   tangent, 842
   undefined, 33
plane angle of a dihedral angle, 634
plane separation postulate, 138
plot, 514
point(s)
   line coordinate of, 76
   plane coordinates of, 511
   space coordinates of, 646
   distance between, 522, 655
   power of with respect to a circle, 871
   representation of, 35
   undefined, 33
point plotting theorem, 116
point-slope form of a linear equation, 569
point of tangency
   of a circle, 830
   of a sphere, 842
polygon(s)
   angles of, 213
   congruence between, 405
   consecutive sides of, 210
   consecutive vericen of, 210
   convex, 211
   regular, 297, 779
   sides of, 209
   similar, 403
   vertex of, 209
polygonal-region(s), 730
   area of, 744
   boundary of, 732
   edges of, 734
   faces of, 734
   interior of, 732
   vertices of, 734
polyhedral angle(s), 788
   edge of, 788
   face of, 789
   face angle of, 789
   vertex of, 788
polyhedron(s), 783
   convex, 784
   edge of, 783
   face of, 783
   regular, 784
   section of, 784
   vertex of, 783
postulate(s), 10
   of Algebra, 57
   of congruence,
      A.S.A., 252
      S.A.S., 251
      S.S.S., 253
   of incidence, 35, 36, 42
   interior of an angle, 174
   parallel, 339
   plane separation, 138
   proportional segments, 412
   protractor, 159
   ruler, 77

power of a point with respect
    to a circle, 871
prime number, 5
prism(s), 796
    altitude of, 799
    base of, 796
    cross-section of, 798
    lateral area of, 799
    lateral edge of, 797
    lateral face of, 797
    lateral surface of, 797
    rectangular, 796
    right, 797
    right-section of, 798
    total area of, 799
    triangular, 796
prismatic surface, 797
product property of
    proportion, 399
projection,
    of a point into a plane, 630
    of a point on a line, 428
    of a segment on a line, 428
    of a set of points into a
        plane, 631
    of a vector, 720
proof, 17
    finding of, 271
    indirect method, 325
    paragraph form, 276
    two column form, 244, 276
    using definitions in, 241
    writing of, 260
properties of
    congruence, 233
        for angles, 235
        for segments, 234
        for triangles, 235
    directed segments, 687
    equality, 233, 235
    order, 63
    parallel lines, 347
    parallel planes, 627
    parallelograms, 603
    proportion, 399
    proportionality, 397
    rectangles, 603
    rhombuses, 603
    scalar products, 719
    similar convex polygons, 406
    squares, 603
    trapezoids, 603
    vectors, 707
property of the
    contrapositive, 329
proportion, 399
    properties of, 399
proportional, 393
proportional segments
    postulate, 412
proportionality, 392
    inverse, 766
    properties of, 397
protractor, 151
protractor postulate, 159
pyramid(s), 803
    altitude of, 803
    base of, 803
    frustum of, 805
    regular, 804
    slant height of, 805
    vertex of, 803
Pythagorean theorem, 433
quadrants, 513
quadrilateral(s), 204
    opposite sides of, 213
    opposite vertices of, 213
    sides of, 204
    vertices of, 204
radian, 153
radius
    of a circle, 819, 821
    outer end of, 821,
    of a regular polygon, 779
    of a sector of a circle, 901
ray(s), 84
    antiparallel, 355
    concurrent, 597
    coordinate of, 159
    endpoint of, 84
    initial, 143
    interior of, 90
    opposite, 85
    ordered pair of, 143
    parallel, 355
    slope of, 556
    terminal, 143
ray-coordinate system, 159
real numbers, 56
reasoning,
    deductive, 10
    indirect, 12
    inductive, 5
rectangle, 578
    area of, 748
    properties of, 603
rectangular parallelepiped, 797
rectangular prism, 796
reflex angle, 143
reflexive property
    of congruence,
        for angles, 235
        for segments, 234
        for triangles, 235
    of equality, 233
    of equivalent directed
        segments, 687

reflexive property, (con't.)
  of parallel lines, 347
  of parallel planes, 627
  of proportionality, 397
  of similar convex polygons, 406
regular polygon(s), 297
  apothem of, 779
  area of, 780
  center of, 779
  central triangle of, 779
  radius of, 779
regular polyhedron, 784
regular pyramid, 804
resultant, 709 ex.
rhombus, 578
  area of, 755
  properties of, 603
Riemannian geometry, 340
right angle, 182
right dihedral angle, 635
right prism, 797
right section of a prism, 798
right triangle, 366
rotation, 143
ruler postulate, 77
S.A.A. theorem, 361
S.A.S. postulate, 251
S.A.S. similarity theorem, 421
scalar(s), 683, 703
scalar product, 718
  properties of, 719
secant, 821
secant-secant angle, 862
secant-segment, 870
  external, 870
section of polyhedron, 784
sector of a circle, 901
  arc of, 901
  area of, 901
  radius of, 901
segment(s), 86
  of a circle, 903 ex.
  concurrent, 597
  congruent, 115
  directed, 684
  endpoints of, 86
  interior of, 90
  length of, 114
  midpoint of, 91, 526, 550
  parallel, 350
  slope of, 554
  tangent, 869
semicircle, 848
separation,
  by a line, 138
  by a plane, 139
  by a point, 133
set(s), 19
  convex, 134
  elements of, 19
  empty, 26
  equality of, 20
  intersection of, 24, 534
  null, 29 ex.
  of real numbers, 55
  union of, 25, 534
set-builder notation, 531
side,
  of an angle, 145
  of a line, 138
  of a plane, 139
  of a quadrilateral, 204
  of a triangle, 201
similar polygons, 403
  properties of, 406
skew lines, 316
slant height of a pyramid, 804
slope,
  of a segment, 554
  of a non-vertical line, 556
  of a non-vertical ray, 556
slope-intercept form of a linear equation, 571
space, 36
  coordinate system in, 641
sphere(s), 820
  center of, 820
  chord of, 821
  concentric, 820
  congruent, 822
  diameter of, 821
  exterior of, 841
  great circle of, 821
  interior of, 841
  radius of, 820, 821
  secant of, 821
  tangent to, 842
square, 578
  area of, 749, 756
  properties of, 603
S.S.S. postulate, 253
S.S.S. similarity theorem, 420
"straight angle", 144
subset, 22
  proper, 25 ex.
substitution property, 233
supplement, 189
supplement theorem, 189
symmetric property
  of congruence,
    for angles, 235
    for segments, 234
    for triangles, 235
  of equality, 233
  of equivalent directed segments, 687

- symmetric property, (con't.)
  - of parallel lines, 347
  - of parallel planes, 627
  - of proportionality, 397
  - of similar convex polygons, 406
- tangent to a circle, 830
  - common external, 877 ex.
  - common internal, 877 ex.
- tangent-chord angle, 861
- tangent circles, 835
- tangent plane, 842
- tangent-secant angle, 862
- tangent-segment, 869
- tangent-tangent angle, 862
- terminal point, 684
- terminus, 684
- tetrahedron, 784
- theorem(s), 10
  - A.A. similarity, 422
  - angle construction, 160
  - betweenness-angle, 166
  - betweenness-addition,
    - for points, 240
    - for rays, 240
  - betweenness-coordinate, 109
  - betweenness-distance, 117
  - hypotenuse-leg, 367
  - isoceles triangle, 275
  - origin and unit point, 78
  - point-plotting, 116
  - Pythagorean, 433
    - converse of, 434
  - S.A.A., 361
  - S.A.S. similarity, 421
  - S.S.S. similarity, 420
  - supplement, 189
  - triangle inequality, 378
  - two-coordinate system, 103
  - two-point, 108
- total area of a prism, 799
- transitive property
  - of congruence,
    - for angles, 235
    - for segments, 234
    - for triangles, 235
  - of equality, 234
  - of equivalent directed segments, 687
  - of order, 63
  - of parallel lines, 347
  - of parallel planes, 627
  - of proportionality, 398
  - of similar convex polygons, 406
- transversal, 317
- trapezoid, 593
  - altitude of, 593
  - area of, 758
  - base of, 593
  - base angles of, 593
  - isoceles, 593
  - legs of, 593
  - median of, 593, 758
  - properties of, 603
- triangle(s), 201
  - altitude of, 427
  - angles of, 201
  - area of, 753
  - circumscribed, 905
  - congruent, 229
  - equiangular, 277
  - equilateral, 277
  - exterior of, 203
  - exterior angle of, 292
  - inscribed, 905
  - interior of, 202
  - interior angle of, 292
  - isosceles, 277
  - median of, 289
  - right, 366
  - sides of, 201
  - vertices of, 201
- triangle inequality theorem, 378
- triangular prism, 796
- triangular-region, 730
- trihedral angle, 789
- two-coordinate system theorem, 103
- two-point form of a linear equation, 569
- two-point theorem, 108
- unequal in the same order, 370
- union of sets, 25, 534
- unit, 70
- unit area, 744
- unit-pair, 70
- unit-point, 76
- unit square, 748
- vector(s), 700, 703
  - components of, 703
  - equal, 704
  - geometrical applications, 714
  - length of, 704
  - magnitude of, 704
  - parallel, 705
  - perpendicular, 717
  - product with scalar, 704
  - projection of, 718
  - properties of, 707
  - scalar product of, 718
  - subtraction of, 705
  - sum of, 705
  - zero, 704

vertex
  of a polygonal-region, 734 ex.
  of a polyhedral angle, 788
  of a polyhedron, 783
  of a pyramid, 803
vertex angle of an isosceles
  triangle, 277
vertical angles, 194
vertical lines, 510
x-axis, 510
x-component of a directed
  segment, 690
x-coordinate, 511, 646
xy-plane, 510, 643
xz-plane, 643
y-axis, 510
y-component of a directed
  segment, 690
y-coordinate, 511, 646
yz-plane, 643
z-coordinate, 646
zero-ray, 159
zero vector, 704

The preliminary edition of this volume was prepared at a writing session held at Yale University during the summer of 1961. This revised edition was prepared at Stanford University in the summer of 1962, taking into account the classroom experience with the preliminary edition during the academic year 1961-62.

The following is a list of those who have participated in the preparation of this volume.

James P. Brown, Atlanta Public Schools, Georgia

Janet V. Coffman, Catonsville Senior High School, Baltimore County, Maryland

Arthur H. Copeland, University of Michigan

Eugene Ferguson, Newton High School, Newtonville, Massachusetts

Richard A. Good, University of Maryland

James H. Hood, San Jose High School, San Jose, California

Michael T. Joyce, DeWitt Clinton High School, New York, New York

Howard Levi, Columbia University

Virginia Mashin, San Diego City Schools, San Diego, California

Cecil McCarter, Omaha Central High School, Omaha, Nebraska

John W. Murphy, Grossmont High School, Grossmont, California

William F. Oberle, Dundalk Senior High School, Baltimore County, Maryland

Mrs. Dale Rains, Woodlawn Senior High School, Baltimore County, Maryland

Lawrence A. Ringenberg, Eastern Illinois University

Robert A. Rosenbaum, Wesleyan University, Connecticut

Laura Scott, Jefferson High School, Portland, Oregon

Harry Sitomer, New Utrecht High School, Brooklyn, New York

Guilford L. Spencer, II, Williams College

Raymond C. Wylie, University of Utah